PHYSICAL REVIEW D VOLUME 25, NUMBER 2 15 JANUARY 1982

Abelian Higgs model: A Monte Carlo study

David J. E. Callaway* and Larry J. Carson
Department of Physics, FM-15, University of Washington, Seattle, Washington 98195

(Received 4 September 1981)

The Abelian Higgs model—a U(1) lattice gauge theory coupled to a fixed-length scalar field—is studied using Monte Carlo techniques. The phase diagram of the zero-temperature model on a four-dimensional Euclidean lattice is elucidated for systems with charge-one and charge-two Higgs fields. The order of various transitions is tentatively determined. Possible implications for quasiconfinement of quarks and gluons are also discussed briefly.

I. INTRODUCTION

The Abelian Higgs model provides us with an excellent prototype for the study of more complex models of matter interacting with gauge fields. It consists of a scalar field of fixed magnitude coupled to the field of a massless photon in the usual U(1) gauge-invariant fashion. As is shown below, the model can be thought of as an appropriate large—self-coupling limit of a lattice version of scalar electrodynamics in which the magnitude of the scalar field is allowed to vary. The analysis of the Abelian Higgs model may thus furnish insight into the nature of the more general but noncompact scalar model.

The phase structure of the fixed-magnitude model has been discussed previously, at both zero[1–3] and finite temperature,[4] in the analytically accessible regions of large and small couplings. However, several important questions are beyond the reach of these techniques, and can at present only be studied by numerical methods such as Monte Carlo simulations. Such calculations have been performed for the Abelian Higgs model in three dimensions,[5] as well as the closely related Z_N models in four dimensions.[6] The focus here is on the Euclidean four-dimensional Abelian Higgs model at zero temperature. The nontrivial phase structure of the lattice theory is mapped out by a Monte Carlo analysis, and in some cases the order of the various phase transitions is tentatively determined.

One specific question which can be probed within this framework concerns the qualitative differences of the phase diagrams for different representations of the Higgs fields. The gauge group in this case is U(1). External sources in the defining fundamental representation (i.e., with unit electric charge) are completely screened only by Higgs fields in the fundamental representation (i.e., which also carry unit charge). In the realm of strong coupling and in the absence of these charge-one scalars electric charges are confined. If a unit-charge scalar field is present, however, it becomes energetically favorable to produce Higgs pairs from the vacuum, thereby screening the sources and spoiling confinement.

The result is that the phase diagram of the charge-one Higgs lattice theory is dramatically different from its counterparts with multiply-charged scalar fields. In the latter case, three distinct phases should occur[1–4]—namely, (1) a confinement phase, where the photon is presumably massive and there are no free charges in the spectrum of the theory, (2) a Higgs phase, where the photon is again massive but electric charges experience only short-ranged interparticle forces, and (3) a Coulomb phase, wherein massless photons give rise to the usual interparticle Coulombic force. With the Higgs field in the fundamental representation, however, analysis[1–4] suggests that the Higgs and confinement phases are continuously connected. Both of these phase structures are verified here numerically.

The work presented here may also have some bearing on the possibility of unconfined quarks and gluons at high energies.[7] In order to implement quasiconfinement in the fashion proposed by Ref. 7, the transition from the confining to the broken-symmetry phase in a continuum SU(3) Higgs model must not be first order.[8] Our findings are consistent with a higher-order transition for the analogous transition in the Abelian Higgs model, provided the Higgs scalars are in the fundamental representation. A discussion of this point may be found at the conclusion of the paper.

II. MODEL AND METHODS

A. The model

The Abelian Higgs model is studied here on a 4^4 hypercubic Euclidean lattice with lattice spacing a. A Higgs field $\sqrt{\beta_H}\,\phi_n$ is defined on each site n of the lattice, subject to the constraint

$$|\phi_n|^2=1 \text{ for all } n. \tag{1}$$

Electromagnetic field variables $U_{n\mu}$ are associated with each link connecting site n with site $n+\hat{\mu}$. The internal-symmetry group is U(1), so that ϕ_n and $U_{n\mu}$ are simple phases $e^{i\theta}$, with $-\pi\leq\theta\leq\pi$. The dynamics of this interacting system are governed by the action

$$A=\beta\sum_{\substack{n\mu\nu\\ \mu\neq\nu}}A_{\Box}(n,\mu,\nu)+\beta_H\sum_{n\mu}A_H(n,\mu), \tag{2a}$$

where the plaquette $A_{\Box}$ and scalar actions A_H are given by

$$\begin{aligned}A_{\Box}(n,\mu,\nu)&=\tfrac{1}{2}(1-U_{n,\mu}U_{n+\hat{\mu},\nu}U_{n+\hat{\nu},\mu}{}^{-1}U_{n,\nu}{}^{-1}),\\ A_H(n,\mu)&=\tfrac{1}{2}\,|\,U_{n,\mu}{}^{q}\phi_{n+\hat{\mu}}-\phi_n\,|^2.\end{aligned} \tag{2b}$$

As seen below, the power q is identified with the charge of the Higgs field. The action is locally gauge invariant with respect to the set of transformations

$$\begin{aligned}U_{n\mu}&\to V_{n+\mu}^{\dagger}U_{n\mu}V_n,\\ \phi_n&\to V_n\phi_n,\end{aligned} \tag{3}$$

where $V_n=e^{i\lambda_n}$ are elements of the underlying U(1) gauge group.

The constraint [Eq. (1)] can also be implemented by allowing the magnitude of the scalar field to fluctuate but including a suitable potential term in the action. One possible extension of Eqs. (2) is

$$\begin{aligned}A'=\beta&\sum_{\substack{n\mu\nu\\ \mu\neq\nu}}A_{\Box}(n,\mu,\nu)+\beta_H\sum_{n\mu}A_H(n,\mu)\\ &+\lambda\sum_n(\,|\phi_n|^2-1)^2.\end{aligned} \tag{4}$$

Note that in the limit $\lambda\to\infty$ configurations in which $|\phi_n\,|$ is different from unity are not important.

With the interpretation

$$\begin{aligned}\beta&=\frac{1}{e^2},\\ \sqrt{\beta_H}\phi_n&=a\Phi(x_n),\\ U_{n\mu}&=\exp[iaeA_\mu(x_n)]\end{aligned} \tag{5}$$

in terms of the lattice spacing a and the electric charge e, the naive continuum limit ($a\to 0$) of Eq. (4) produces the familiar Euclidean action of scalar electrodynamics:

$$A'_{\text{continuum}}=\int d^4x[\tfrac{1}{4}F_{\mu\nu}{}^2(x)+\tfrac{1}{2}\,|\,(\partial_\mu+iqeA_\mu)\Phi(x)\,|^2+\lambda'(\,|\Phi(x)|^2-f)^2] \tag{6}$$

with

$$\lambda'\equiv\frac{\lambda}{\beta_H{}^2} \tag{7a}$$

and

$$f\equiv\frac{\beta_H}{a^2}. \tag{7b}$$

Here the parameter q specifies the charge of the Higgs field Φ in units of e. Extracting the continuum limit in this manner, however, is invalid as the simple $a\to 0$ limit is singular. Instead, the bare coupling constants (e,β_H,λ) must be adjusted with the removal of the ultraviolet cutoff to realize finite limiting values for all physical quantities. Such a procedure may only be possible for theories whose naive continuum limits correspond to renormalizable theories.[9] It is here we see the advantage of casting the lattice model in the form of Eq. (4), whose naive continuum limit [Eq. (6)] is a renormalizable theory of substantial interest.[10] The technique of adjusting the bare coupling constants together with the lattice spacing may be applicable in this case. It is not clear that the analogous statement may be made for the Abelian Higgs model [Eqs. (2)] whose naive $a\to 0$ limit gives a generalization of the nonlinear σ model, which is nonrenormalizable.

Therefore, no claim is made that the phase structure numerically examined in this work has a direct connection to any continuum theory; instead, the results here represent an interesting and necessary elaboration of the more realistic theory of Eq. (4) in the limit of large coupling ($\lambda \to \infty$). To ascertain the extent to which this statement is true, work on the lattice model of Eq. (4) at finite values of λ is now in progress.

B. Detection of phase transitions

The vacuum expectation value (VEV) of any function F of the field variables ϕ_n and $U_{n\mu}$ can be evaluated numerically for various values of β and β_H. These expectation values are defined by the relation

$$\langle F(\phi_n, U_{n\mu})\rangle_0 \equiv Z^{-1} \int \mathscr{D}[\phi_n]\mathscr{D}[U_{n\mu}]e^{-A}F(\phi_n, U_{n\mu}), \tag{8}$$

where the partition function Z is given by

$$Z = \int \mathscr{D}[\phi_n]\mathscr{D}[U_{n\mu}]e^{-A}. \tag{9}$$

In Eq. (8) $\mathscr{D}[e^{i\alpha}]$ is the group-invariant measure for representation $e^{i\alpha}$ of the U(1) gauge group and is a simple integration over the phase α:

$$\int \mathscr{D}[e^{i\alpha}] = \prod_n \left[\int_{-\pi}^{\pi} \frac{d\alpha_n}{2\pi}\right]. \tag{10}$$

Note that the gauge volume is finite on the lattice. Thus there is no need to fix the gauge; indeed general experience with Monte Carlo simulations suggests better convergence if the gauge is allowed to fluctuate.[11]

The technique of Monte Carlo computer simulation is to make the approximation

$$\langle F(\phi_n, U_{n\mu})\rangle_0 \simeq \frac{1}{N}\sum_{i=1}^{N} F(\phi_n^{(i)}, U_{n\mu}^{(i)}), \tag{11}$$

where (i) labels a sequence of field configurations, each chosen according to a specified algorithm. For an algorithm satisfying the requirements of detailed balance and positive definiteness, this approximation of the full path integral can be shown to obey a central limit theorem.[12] This theorem assures us that the approximation converges to $\langle F\rangle_0$ in the limit that N, the total number of configurations, goes to infinity. The calculations presented here utilize the Metropolis algorithm,[13] which fulfills these requirements.

Because the Abelian Higgs model possesses a continuous U(1) symmetry, Elitzur's theorem[14] states that no local order parameter exists. This theorem does not preclude the determination of the phase structure, however. Nonlocal gauge-invariant quantities such as the average plaquette

$$P = \frac{1}{N_\square V}\sum_{\substack{n\mu\nu \\ \mu\neq\nu}} \langle A_\square(n,\mu,\nu)\rangle_0 \tag{12}$$

and the average scalar

$$S = \frac{1}{N_H V}\sum_{n\mu} \langle A_H(n,\mu)\rangle_0 \tag{13}$$

(where V is the number of sites in the system, $N_\square$ the number of plaquettes per link and N_H the number of links per site) are accessible to measurement. The bulk of this work consists of a calculation of P and S as functions of q, β, and β_H.

Lines of phase transitions in the (β,β_H) plane are most easily found via a hysteresis effect. Figure 1 illustrates the method: we begin at large values of β in an ordered configuration (all ϕ_n, $U_{n\mu}$ set equal to unity) and step down in β, measuring the average plaquette P for a finite number $M(\lesssim 30)$ of Monte Carlo iterations at each value of β selected. Far away from the critical point, the time taken for the system to settle close to equilibrium is small ($\lesssim 5$ iterations) compared to M.[15] In the vicinity of the critical point, however, the relaxation

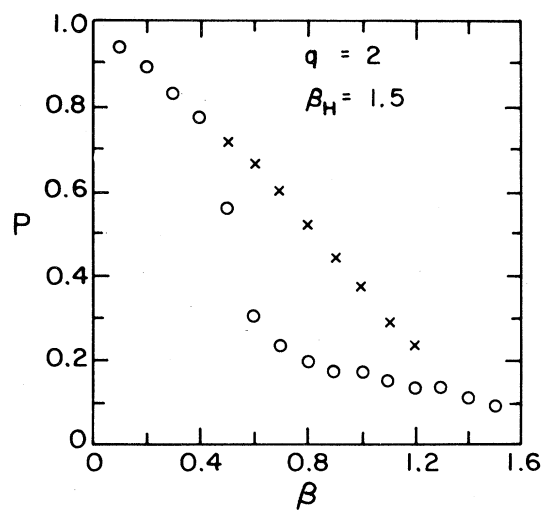

FIG. 1. Detection of a phase transition via a hysteresis effect. The circles (○) denote "heating" runs starting from large β and decreasing stepwise to smaller values. The crosses (×) indicate "cooling" runs performed after the lowest value of β is reached. The value of β where the width of the hysteresis in P is a maximum is taken to be the critical value β_c.

time exceeds M and equilibrium is not reached. In graphical terms, the initially ordered configuration has insufficient time to disorder, resulting in depressed values of the average plaquette.

For sufficiently low values of β, the system is in a totally disordered phase. At this point we reverse our steps and increase β back to its initial value. This time the increase in relaxation time near the critical point will tend to suspend P at higher values. The net result is that near the critical β a hysteresis effect is observed. It is precisely this characteristic of a general phase transition that serves as the signal of the phase transition. As an operational definition, the critical point β_c is taken to be the point where the width in P (or S, if β_H is varied) of the hysteresis curve is a maximum.

Among other questions of interest is the order of the various transitions, which as already indicated may have some bearing on the possibility of quark/gluon quasiconfinement. In principle Monte Carlo calculations permit a distinction to be made between first- and higher-order transitions.[6,16] The technique is to examine the action at the critical line as a function of iteration from initially ordered and random configurations (see Fig. 2). If the transition is first order the two adjoining phases (each characterized by different values of the average action) are metastable at the critical point. In contrast, a higher-order transition possesses no such discontinuity at the phase boundary. Given a sufficient number of iterations, the action in this latter case will settle to a value independent of the initial configuration. Such a procedure is difficult in practice, however, because of statistical errors. Thus only *tentative* conclusions about the order of the phase transitions are given.

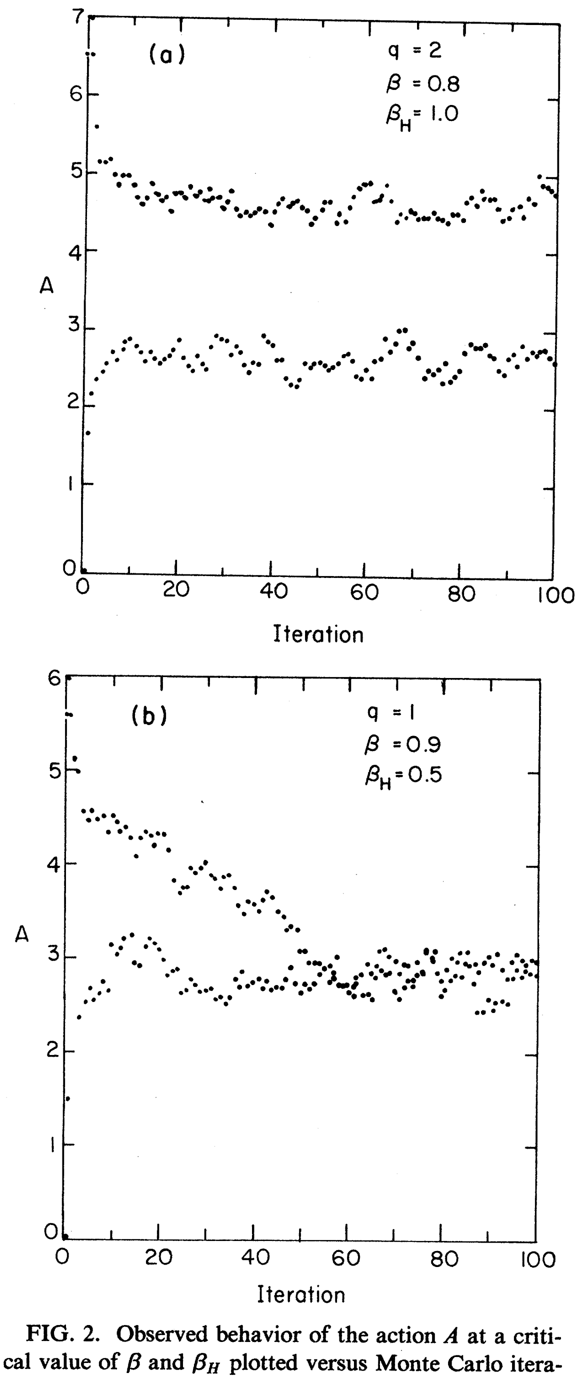

FIG. 2. Observed behavior of the action A at a critical value of β and β_H plotted versus Monte Carlo iteration. The system is initially placed in an ordered or random configuration. (a) Portrays first-order behavior; (b) higher-order behavior.

III. RESULTS

Figures 3(a) and 3(b) show the phase diagrams for charge-one and charge-two Higgs scalars, determined by the hysteresis effect discussed in the previous section. All work is performed on a 4^4 Euclidean lattice with periodic boundary conditions.

In the region of small β_H ($\lesssim 0.6$), the two cases are similar in structure, although not numerically identical. At $\beta_H=0$, the scalar field decouples and the pure U(1) gauge theory is retrieved. This theory has a single critical point at

$$\beta=0.98\pm0.01, \quad \beta_H=0, \tag{14}$$

a result which coincides with more extensive computations.[17] (The errors quoted in this work are subjective estimates.) The observed transition persists for small values of β_H until a triple point at

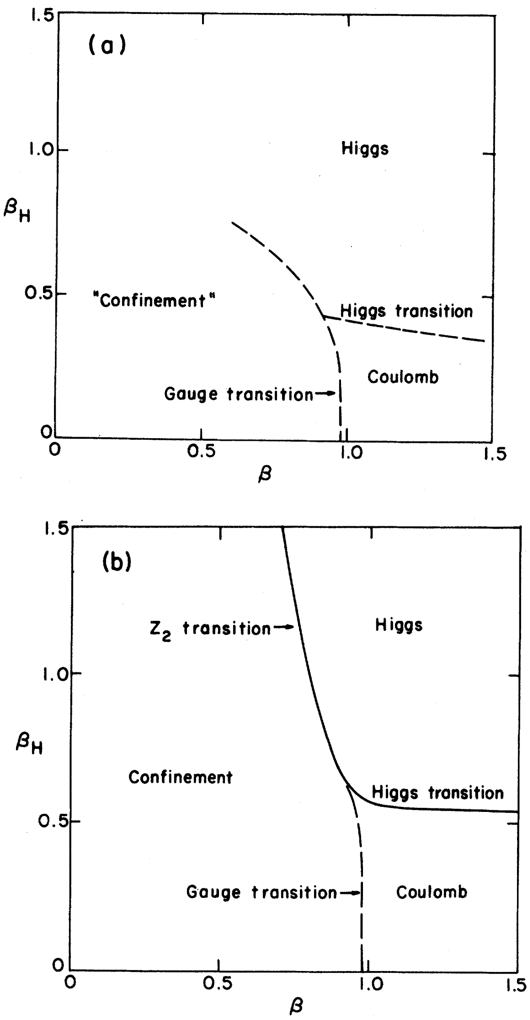

FIG. 3. (a) Phase diagram for the $q=1$ Abelian Higgs model. First-order transitions are drawn with solid lines, higher-order transitions are drawn with broken lines. (b) Phase diagram for the $q=2$ Abelian Higgs model. The legend is as for (a).

$$\begin{aligned}(q=1)\quad &\beta=0.92\pm0.05,\\ &\beta_H=0.43\pm0.05,\\ (q=2)\quad &\beta=0.92\pm0.05,\\ &\beta_H=0.60\pm0.05\end{aligned}\tag{15}$$

is reached. The transitions along this line seem higher than first order for both values of the Higgs charge q.

Two additional boundaries extend outwards from this triple point. For both the charge-one and charge-two cases, a line of (Higgs) transitions extends to the right for all values of β probed. This result is in accord with theoretical expectation;[1-4] the low-β_H side presumably represents the Coulomb (massless photon) phase of the theory. If this identification is indeed correct, then in spite of a nonzero vacuum expectation value for the squared magnitude of the Higgs field, quantum fluctuations apparently restore the symmetry of the vacuum in this region. By the methods described above, this transition is found to be first order for the charge-two case. Surprisingly enough, the same methods tentatively determine the Higgs transition to be higher order for the charge-one case for all values of β (≤ 1.6) examined.

In Ref. 1 and references contained therein the Higgs transition is predicted to be first order for sufficiently large but finite β. Thus the order of the Higgs transition may change for large β in the $q=1$ case.

The other line of phase transitions leaving the triple point also displays a distinctly different behavior for the two phase diagrams. For the charge-one case, the line of apparent higher-order phase transitions ends at the critical point:

$$(q=1)\quad \beta=0.6\pm0.1,\quad \beta_H=0.75\pm0.05. \tag{16}$$

Thus, as stated in the Introduction, the Higgs and "confinement" phases of the theory are found to be continuously connected.

The same transition line in the $q=2$ case does not appear to terminate. The confinement and Higgs phases remain distinct for all values of β_H probed. The line of transitions appears first order, as expected, since in the limit of large β_H the theory approaches a Z_2 gauge theory, which possesses a first-order transition.[6]

The existence of a deconfining transition was also confirmed by calculating the expectation value of the Wilson line, defined as the product of all timelike links with spatial coordinates $\vec{n} = (n_1,n_2,n_3)$:

$$L(\vec{n})=\mathrm{Re}\left[\prod_{n_4=1}^{N_4} U_{(\vec{n},n_4);\hat{4}}\right], \tag{17}$$

where N_4a is the length of the lattice in the (Euclidean) time direction. Elementary manipulations[18] demonstrate that

$$\langle L(\vec{n})\rangle_0=\exp[-N_4aF], \tag{18}$$

where F is the free energy of an isolated electron.

As pointed out in Ref. 1, the Wilson line retains validity as an order parameter for the multiply-charged theory. Results for this quantity in the $q=2$ case indicate that $\langle L(\vec{n})\rangle_0$ is consistent with zero in the confinement phase.

IV. CONCLUSIONS

The phase diagrams for the charge-one and charge-two Higgs fields coupled to a U(1) lattice gauge theory have been determined on a 4^4 Euclidean lattice at zero temperature. Connections with other such lattice models, as well as with continuum theories, have been discussed. The order of the various transitions present in the two models have also been tentatively classified as first or higher order.

Perhaps the most closely related lattice models incorporate Z_N as the underlying gauge group. This relation follows from the simple observation that a Z_N symmetry becomes a U(1) symmetry in the limit $N\rightarrow\infty$. Thus the general nature of the U(1) and Z_N phase diagrams are expected to be very similar disregarding those features which disappear as $N\rightarrow\infty$. Indeed, Monte Carlo results[6] for Z_2 and Z_6 lattice gauge theories coupled to Higgs in the fundamental representation (analogous to the $q=1$ Higgs case), as well as for a Z_6-gauge—Z_3-Higgs system (analogous to $q=2$) are in qualitative accord with the results given here.

As stated in the Introduction, our results may have relevance for the possibility of quasiconfinement,[7] which may occur if the QCD Higgs-confinement transition is higher than first order. The results given here tentatively indicate quasiconfinement may occur in a coupled QCD-Higgs system given a number of provisos: (1) the Higgs field lies in the fundamental representation of the underlying gauge group, (2) the order of the Higgs-confinement transition is independent of the gauge group, (3) the constraint $|\phi_n|^2=1$ is not too serious a distortion of variable-magnitude scalar electrodynamics [Eq. (4)]; and (4) the transition remains higher order in the continuum limit.

Points (2) and (3) are ideally suited for further Monte Carlo exploration. For example, recent calculations[19] have been performed for a non-Abelian gauge—Higgs system based on the icosahedral subgroup of SU(2). No evidence for a Coulomb phase was found, and the deconfining transition for the fundamental representation was found to be higher order, as here for the U(1) case. However, the deconfining transition for the case of Higgs in the adjoint representation was also suggestive of a higher-order transition. This last result contrasts with the first-order transition found in this work for the (corresponding) charge-two Higgs case. Further work on non-Abelian systems is needed to clarify this difference.

Note added. While this manuscript was in preparation, the authors learned of another Monte Carlo calculation of the Abelian Higgs model on an 8^4 lattice.[20] The phase diagrams from that calculation agreed quantitatively with results given here.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the assistance of L. McLerran in instigating this study, as well as many stimulating discussions with M. Baker and S. D. Ellis. Conversations with L. S. Brown, R. Cahn, M. Einhorn, D. R. T. Jones, B. Petersson, E. K. Riedel, and D. J. Thouless were also very helpful. This work was supported in part by the U. S. Department of Energy.

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*Present address: High Energy Physics Division, Argonne National Laboratory, Argonne, Illinois 60439.

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