Effective potential of lattice ϕ^4 theory

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The structure of the effective potential for lattice ϕ^4 theory is discussed. It is shown that the effective potential $V(\hat{\phi})$ is undefined in an infinite lattice system for certain values of $\hat{\phi}$ if spontaneous symmetry breaking occurs. It is also shown that $d^2V/d\hat{\phi}^2 \geq 0$ for all ϕ, thus precluding the familiar "double-well" shape suggested by the classical potential. These results do not depend on the spacetime dimensionality of the lattice or upon the particulars of any loop expansion. A graphical approximation procedure for the effective potential is formulated and compares very favorably with Monte Carlo results. Comparisons with strong-coupling-expansion predictions are also made.

A careful analysis of the vacuum structure of a field theory is a necessary precursor to understanding its physical content. Typically this analysis proceeds by calculating (via perturbation theory) a quantity known as the effective potential,[1-4] the minimum of which furnishes information as to the nature of the lowest energy eigenstate of the theory. In order to determine the effective potential accurately, however, it is important that nonperturbative effects be considered as well. One convenient way to discuss such effects is to perform this analysis on a discrete version of the theory.[5] Lattice field theories possess a nonperturbative regulator—the lattice spacing—and are amenable to numerical analysis by Monte Carlo methods. The simplicity and directness of this approach is demonstrated here by determining the effective potential of lattice ϕ^4 theory.

Naive expectations notwithstanding, it is possible to show that spontaneous symmetry breaking in this theory is *not* accompanied by the presence of a double well in the effective potential. In fact, the second derivative of the effective potential $V(\hat{\phi})$ with respect to $\hat{\phi}$ can never be negative, thus precluding the familiar double-well shape for $V(\hat{\phi})$. The presence of a nontrivial vacuum in lattice ϕ^4 theory instead implies[4] the *nonexistence* of the effective potential for a range of $\hat{\phi}$. Implications of this result include interesting analogies with thermodynamics and a novel graphical approximation procedure reminiscent of the Maxwell construction. This graphical approximation agrees with a Monte Carlo calculation and in certain instances is an improvement over results obtained from the strong-coupling expansion.

The continuum limit of such lattice theories must be considered with caution. For example, the continuum limit of lattice ϕ^4 field theory is known to be noninteracting (i.e., the renormalized quartic coupling constant is zero) in greater than three dimensions.[6-9] On the other hand, ϕ^4 theory in three spacetime dimensions is known to have an interacting continuum limit.[10] The graphical approximation appears to be equally valid for lattice theories in three and four dimensions and is thus presumably applicable to *interacting* continuum theories.

At this point it is useful to review the effective-potential formalism for ϕ^4 theory on a Euclidean lattice of N sites. In this formalism the generating functional $W\{J\}$ is defined implicitly by

$$e^{W\{J\}} \equiv \int \mathscr{D}\phi e^{-S\{\phi,J\}} , \tag{1a}$$

where

$$S\{\phi,J\} \equiv \tfrac{1}{2} \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2 + \sum_i [U(\phi_i) + J_i\phi_i] , \tag{1b}$$

and

$$U(\phi_i) \equiv \lambda(\phi_i{}^2 - f)^2 , \quad \int \mathscr{D}\phi = \prod_i \int d\phi_i . \tag{1c}$$

The notation $\langle ij \rangle$ refers to all nearest-neighbor pairs on the lattice (summed once). The $\{\phi\}$ are real fields ranging from $-\infty$ to ∞.

From Eqs. (1) it follows that the expectation value of ϕ_i in the presence of sources $\{J\}$ is

$$\hat{\phi}_i \equiv \langle \phi_i \rangle_J = -\frac{dW\{J\}}{dJ_i} . \tag{2}$$

The effective action $\Gamma\{\hat{\phi}\}$ is given by the Legendre transform of $W\{J\}$, i.e.,

$$\Gamma\{\hat{\phi}\} \equiv W\{J\} - \sum_{i=1}^{N} \langle \phi_i \rangle_J J_i . \tag{3}$$

The right-hand side of Eq. (3) is rendered an explicit function of the $\{\hat{\phi}\}$ by using Eq. (2). The effective potential is determined from Eq. (3) by setting all the $\{\hat{\phi}\}$ equal to some value $\hat{\phi}$,

$$V(\hat{\phi})=\frac{1}{N}\Gamma\{\phi_i=\hat{\phi}\} \ . \tag{4}$$

Equations (2) and (3) imply a useful result which is dual to Eq. (2), namely,

$$\frac{d\Gamma\{\phi\}}{d\hat{\phi}_i}=-J_i \ . \tag{5}$$

From Eq. (5) and Ref. 3 it follows that the presence of a nontrivial vacuum in the theory (i.e., $\lim_{J\to 0}\langle\phi\rangle_J\neq 0$) corresponds to a minimum of the effective potential.

Equations (1), (2), and (5) can be used to define a simple procedure for evaluating the effective potential in lattice ϕ^4 theory. First the expectation value $\langle\phi\rangle_J=\hat{\phi}$ for uniform $\{J\}=J$ is calculated by numerical or other means. This result is inverted to obtain $J(\hat{\phi})$. Equation (5) can then be integrated to obtain $\Gamma(\hat{\phi})$ and hence the effective potential $V(\hat{\phi})$.

The pivotal significance of the quantity $\langle\phi\rangle_J$ within this procedure demands that its analytic behavior be studied closely. Several elementary though important properties of this quantity which crucially delimit the behavior of the effective potential are listed below.

(i) The expectation value $\langle\phi\rangle_J$ is antisymmetric in J, i.e.,

$$\langle\phi\rangle_J=-\langle\phi\rangle_{-J} \ , \tag{6}$$

and $\langle\phi\rangle_J=0$ when $J=0$. These results follow directly from Eqs. (1). Furthermore, for any range of parameters where the vacuum of the theory described by Eqs. (1) is nontrivial it must be true that

(ii)

$$\begin{aligned}\langle\phi\rangle_J&\to\zeta \text{ as } J\to 0^- \\ &\to-\zeta \text{ as } J\to 0^+ \ ,\end{aligned} \tag{7}$$

for some $\zeta>0$. Note that as λ increases without bound ζ^2 approaches f.

(iii) By differentiating Eq. (2), it can be shown that for uniform J

$$\frac{-d\langle\phi\rangle_J}{dJ}=\sum_i\langle(\phi_i-\langle\phi_i\rangle_J)^2\rangle_J\geq 0 \ , \tag{8a}$$

and thus that

$$\frac{d^2V(\hat{\phi})}{d\hat{\phi}^2}=-\frac{dJ(\hat{\phi})}{d\hat{\phi}}\geq 0 \ , \tag{8b}$$

wherever this derivative is defined.

(iv) For large J (i.e., a large applied current) the classical result

$$J_{\text{class}}(\hat{\phi})=-4\lambda\hat{\phi}(\hat{\phi}^2-f) \ , \tag{9}$$

should be a good approximation to $J(\hat{\phi})$. Note that for $\hat{\phi}^2<f/3$, $J_{\text{class}}(\hat{\phi})$ violates property (iii).

(v) The expectation value $\langle\phi\rangle_J$ is a single-valued function of J. This property is perhaps best seen on a finite lattice. There the path integral of Eqs. (1) consists of a finite number of integrations of a bounded and singularity-free integrand. This expectation value of course remains single-valued in the limit of an infinite system.

These five properties of $\langle\phi\rangle_J$ imply that the current $J(\hat{\phi})$ for an infinite system must have the qualitative form depicted by the heavy line plotted in Fig. 1(a) if the reflection symmetry $\phi\to-\phi$ is spontaneously broken by the ground state. Note that $\langle\phi\rangle_J$ as a function of J is discontinuous; i.e., the relation Eq. (2) cannot be inverted for all $\langle\phi\rangle_J$. Thus for an infinite system the derivative of the functional $W(J)$ with respect to J is discontinuous. For a finite system, on the other hand, $W(J)$ must be an analytic function of J. As no phase transition can occur in such a case, property (ii) cannot hold for finite systems. The plot of $J(\hat{\phi})$ versus $\hat{\phi}$ for this system is therefore of the form given by the dashed curve in Fig. 1(a). The classical value of the current $J_{\text{class}}(\hat{\phi})$ is plotted in Fig. 1(b).

An analogy can be made between the curves shown in Fig. 1 and the isotherms ("P-V diagram") of a classical thermodynamic system undergoing a phase transition.[11] The analogy is constructed by associating the current J and the expectation value $\langle\phi\rangle_J$ of the field-theoretic system with the pressure

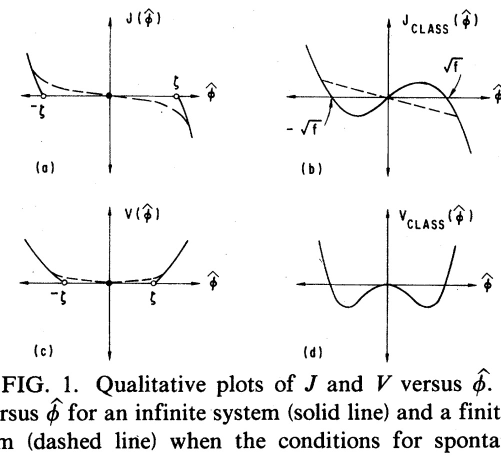

FIG. 1. Qualitative plots of J and V versus $\hat{\phi}$. (a) J versus $\hat{\phi}$ for an infinite system (solid line) and a finite system (dashed line) when the conditions for spontaneous symmetry breaking in the infinite system are met. Note that $J=0$ at $\hat{\phi}=0$ in both cases. (b) J_{class} versus $\hat{\phi}$ when $f>0$ (solid line). Dashed line is discussed in the text. (c). Effective potentials associated with each of the currents shown in (a). The legend is as above. (d) Classical form of the effective potential corresponding to the current $J_{\text{class}}(\hat{\phi})$ plotted in (b).

P and volume V, respectively, of the thermodynamic system. Pursuant to this analogy the exact current, $J(\hat{\phi})$, is associated with the pressure as a function of volume, $P(V)$, calculated from a partition function while the current $J_{\rm class}(\hat{\phi})$ is identified with $P(V)$ calculated from some phenomenological equation of state (such as the van der Waals equation). Within the framework of the grand canonical ensemble, it can be shown that the compressibility $-\partial P/\partial V$ cannot be negative and that V is discontinuous at the phase transition. These results correspond to property (iii) and the discontinuity of $\hat{\phi}$ about $J=0$ (cf. Fig. 1), respectively.

The above analogy suggests a simple graphical prescription for obtaining a useful approximation of $J(\hat{\phi})$ from $J_{\rm class}(\hat{\phi})$. This prescription is reminiscent of the Maxwell construction[12] of thermodynamics and proceeds as follows. For $f \leq 0$, when $J_{\rm class}(\hat{\phi})$ satisfies the positivity property Eqs. (8) and does not vanish except at the origin, $J_{\rm class}(\hat{\phi})$ itself is taken as the required approximation. For $f>0$, two cases are to be distinguished. In the case of an *infinite* lattice, the unphysical portion of $J_{\rm class}$, where $\hat{\phi}^2 \leq f$, is replaced by a point at the origin. For a *finite* lattice—required for Monte Carlo calculations—the approximation must be modified because of the aforementioned analyticity of $W[J]$. This modification consists of replacing the unphysical segment of $J_{\rm class}(\hat{\phi})$ with a line of small negative slope passing through the origin, as shown by the dashed line of Fig. 1(b). The slope of this line tends to zero in the limit of large system size; hence in practice a horizontal line suffices.[13]

The validity of this rather appealing graphical construction can be studied by a numerical computation of $\langle \phi \rangle_J$. Figures 2—5 present the results of a Monte Carlo calculation of this quantity for a 4^4 lattice. An important feature of this calculation is the method used to "update" the lattice in order to bring it into equilibrium. Each ϕ_i is updated by first generating a new field $\phi_i^{\rm new}$ given by

$$\phi_i^{\rm new} = \phi_i + (2r-1)\Delta \; ,$$

where r is a random number uniformly distributed between zero and unity and Δ is a parameter chosen empirically ($\Delta \sim 1-10$ in the present calculation). Acceptance of the generated value $\phi_i^{\rm new}$ is governed by the Metropolis algorithm.[14] Undesirable correlations are avoided by only measuring each $\langle \phi \rangle_J$ after ten updates of the entire lattice have been completed. Also, the entire lattice is first allowed to equilibrate for 100 iterations before measurements are taken.

Figures 2 and 3 display typical results of Monte Carlo calculations of $J(\hat{\phi})$ as a function of $\hat{\phi}$. A comparison of these plots with those of Fig. 1

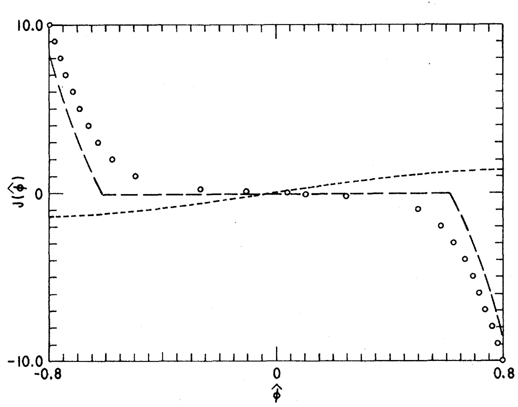

FIG. 2. Monte Carlo-generated values of $J(\hat{\phi})$ (data points) for a *finite* lattice. Strong-coupling approximation of $J(\hat{\phi})$ (short dashes). Graphical approximation of $J(\hat{\phi})$ discussed in the text (long dashes). In each case $\lambda=10$ and $f=0.375$.

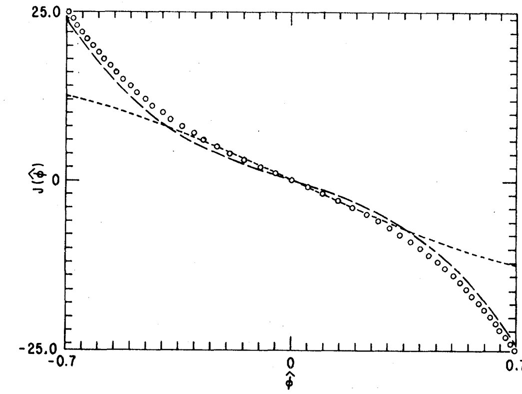

FIG. 3. Monte Carlo-generated values of $J(\hat{\phi})$ (circles) compared with strong-coupling-expansion result (short dashes). Classical current $J_{\rm class}$ (long dashes). In each case $\lambda=10$ and $f=-0.375$.

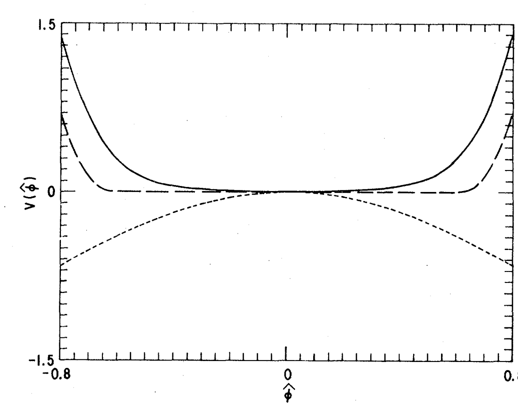

FIG. 4. Effective potentials associated with the currents of Fig. 2 for *finite* lattices. The legend is as before.

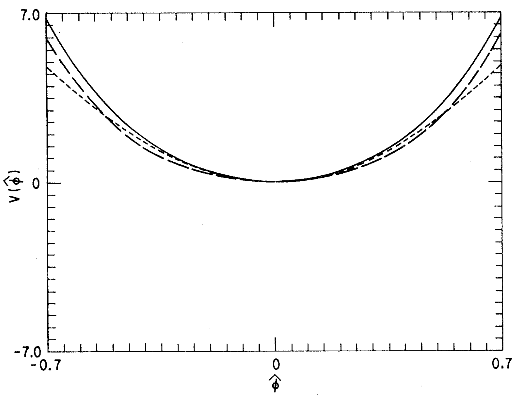

FIG. 5. Effective potentials associated with the currents of Fig. 3. The legend is as before.

suggests the presence (absence) of a nontrivial vacuum for ϕ^4 theory on an infinite lattice for $f>0$ ($f<0$). The strong-coupling-expansion results of Ref. 7 and the results of the graphical approximation discussed above are also shown. In Figs. 4 and 5, the effective potentials corresponding, respectively, to the currents of Figs. 2 (apparently broken symmetry) and 3 (unbroken symmetry) are shown. The effective potentials for the Monte Carlo-generated current were obtained by fitting polynomials to the data of Figs. 2 and 3 and integrating the result analytically.

The comparison of the strong-coupling expansion with the graphical approximation and Monte Carlo results presented above warrants a more careful analysis. The strong-coupling expansions of $V(\hat{\phi})$ given in Ref. 7 are an expansion in powers of $\hat{\phi}$. The expansion is thus continuous everywhere, and in its presumed region of validity—the limit of large λ—is expected to converge only for small $\hat{\phi}$. Thus in Fig. 3 (where spontaneous symmetry breaking does *not* occur) the strong-coupling expansion is a better approximation to the Monte Carlo results than the graphical construction for small $\hat{\phi}$, but it is a poorer approximation for larger values of $\hat{\phi}$. When spontaneous symmetry breaking *does* occur the strong-coupling expansion must fail, as $V(\hat{\phi})$ in an infinite system only exists (except for a single point at the origin) for $\hat{\phi}^2>\zeta^2$.

It should be emphasized at this point that all the qualitative arguments presented here as to the nature of lattice ϕ^4 theory (and quantitatively confirmed above by Monte Carlo calculation) are independent of the dimensionality of the lattice. The Monte Carlo calculations presented here were also repeated for three-dimensional systems with similar results. Thus the known triviality of the continuum limit of lattice ϕ^4 theory in four dimensions (and the corresponding known *non*triviality of the three-dimensional theory) discussed above play no role in this analysis.

The conclusions presented above are valid for a ϕ^4 field theory involving a single scalar field. The applicability of ideas such as the graphical approximation developed above to more complicated systems is under study.[15] One difference is that the positivity condition on the gradient of $\langle\phi\rangle_J$, property (iii), for a field theory involving a multicomponent scalar field $\vec{\phi}$ does not directly imply the monotonicity of the associated components of the current J. In particular for a two-component field theory it can be shown that

$$\left[\frac{\partial\langle\phi_1\rangle_J}{\partial J_1}\right]_{J_2}\geq 0\,, \tag{10}$$

where the notation implies that the first component J_1 of the current is varied while the second component J_2 is held fixed. Equation (10) does *not* imply however that

$$\left[\frac{\partial J_1}{\partial\hat{\phi}_1}\right]_{\hat{\phi}_2}\geq 0\,. \tag{11}$$

Certain features of the above analysis should however have universal validity.[15] For example, the form of the effective potential discussed above is essentially implied by the thermodynamics of phase transitions and should therefore be qualitatively valid for more complicated (e.g., multicomponent) theories. The inadequacies of the strong-coupling expansion alluded to above are presumably manifested by any perturbative calculation (such as a loop expansion), for such expansions cannot faithfully reproduce the critical behavior of the effective potential.

The latter point can be illustrated by considering the case of a complex scalar field interacting electromagnetically. If the real and imaginary parts of the scalar field are denoted by ϕ_1 and ϕ_2, then in a given gauge it can be shown that the Jacobian

$$\mathscr{J}\equiv\frac{\partial(J_1,J_2)}{\partial(\hat{\phi}_1,\hat{\phi}_2)}\geq 0\,, \tag{12}$$

is never negative. The current $\vec{J}=(J_1,J_2)$ is the generalization of the single-component current J discussed above. The relation Eq. (12) can be considered an extension of property (iii) above.

Coleman and Weinberg give[1] the result of a one-loop calculation of the effective potential of this theory in the continuum:

$$V(|\hat{\phi}|)=A|\hat{\phi}|^4\left[\ln\left[\frac{|\hat{\phi}|^2}{\Lambda^2}\right]-\tfrac{1}{2}\right]. \tag{13}$$

Here A is a numerical constant and Λ gives the location of the minimum of V. The Jacobian Eq. (12)

for this case is given by

$$\mathscr{J}=16A^2\left[\frac{|\hat{\phi}|^2}{\Lambda^2}\right]\left[3\ln\left[\frac{|\hat{\phi}|^2}{\Lambda^2}\right]+2\right]\ln\left[\frac{|\hat{\phi}|^2}{\Lambda^2}\right]$$

$$=\frac{d^2V(|\hat{\phi}|)}{d|\hat{\phi}|^2}\frac{dV(|\hat{\phi}|)}{d|\hat{\phi}|}\,. \tag{14}$$

For $\Lambda^2 e^{-2/3}<|\hat{\phi}|^2<\Lambda^2$, the Jacobian in Eq. (14) violates the positivity constraint Eq. (12). Of course (as is pointed out in Ref. 1) higher order terms in the loop expansion involve more powers of $\ln(|\hat{\phi}|^2/\Lambda^2)$, so it is not surprising that the loop expansion fails for $|\hat{\phi}|^2<\Lambda^2$. See also Ref. 15.

The results of the present analysis are now summarized. It has been shown above that the effective potential $V(\hat{\phi})$ for lattice ϕ^4 theory on an infinite lattice does not exist for a range of $\hat{\phi}$ near the origin if spontaneous symmetry breaking occurs. This result is independent of the spacetime dimensionality of the system. A novel graphical procedure (reminiscent of the thermodynamic Maxwell construction) was proposed and used to approximate the effective potential. The results of this approximation procedure compare favorably with Monte Carlo results. A strong-coupling expansion of the effective potential was also studied. This approximation does not always satisfy certain general properties of ϕ^4 lattice field theory. Nevertheless, in its expected region of validity it is consistent with the Monte Carlo calculation when spontaneous symmetry breaking does *not* occur.

Research upon the possibility of generalizing the results of this paper to more complicated systems (e.g., the Abelian Higgs model[16]) is currently underway. Perhaps the most interesting result of this paper—the nonexistence of the effective potential for a range of $|\hat{\phi}|$ when spontaneous symmetry breaking occurs—appears to be a very general phenomenon.[15]

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[12]Phenomenologically derived equations of state (such as the van der Waals equation) may give rise to unphysical regions of negative compressibility in the phase diagram. An *ad hoc* graphical procedure—the Maxwell construction—can be used to improve the approximation. This construction effectively replaces the unphysical portion of the isotherm, analogous to the region of Fig. 1(b) where $\hat{\phi}^2\leq f$, with a horizontal line. See *The Scientific Papers of James Clerk Maxwell,* edited by W. D. Niven (Dover, New York, 1890), Vol. II, p. 424.

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