Random matrices, fractional statistics, and the quantum Hall effect

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The fractional-statistics Laughlin picture of the quantum Hall effect is reformulated as a random-matrix problem. This reformulation connects two large sets of results, and should lead to simplifications for both analytical and numerical studies.

Random-matrix models have recently been employed as theories of strings and quantum gravity.[1,2] The basis for this connection is the observation[2] that the problem of diagram enumeration in random-matrix models is equivalent to that of constructing surfaces by assembling together flat elementary polygons. This association makes it possible easily to construct quantities like correlation functions and to apply powerful combinatorial methods. It also permits more efficient computer algorithms to be formulated.

Interestingly enough, there is a connection between random-matrix models and those[3] of the fractional quantum Hall effect (FQHE). This is not too surprising, since the FQHE involves topological objects (magnetic-flux quanta) in a plane. It has been noticed[3] previously that electrons in the FQHE behave much like a Coulomb gas in a harmonic potential, which in turn is a property of the eigenvalues of random matrices.[4,5] Similarities between the $S=\frac{1}{2}$ Heisenberg antiferromagnet and random matrices have also been observed.[6] In what follows the connection between random matrices and the (fractional-statistics) Laughlin model of the FQHE will be made explicit by a direct construction.

A central feature of Laughlin's picture of the FQHE is a set of wave functions of N electrons in a magnetic field $\mathbf{B}$, each located at plane coordinates $z_j=x_j+iy_j$, with $j=0,1,\ldots,N-1$. These wave functions describe circular droplets of an incompressible quantum fluid. In the symmetric gauge $\underset{\sim}{\mathbf{A}}=\frac{1}{2}\mathbf{B}\times\underset{\sim}{\mathbf{r}}$ these wave functions are (apart from a normalization constant):

$$|m\rangle=\psi_m(z_0,\ldots,z_{N-1})=(\Delta_N)^m\exp\left[-\frac{1}{4}\sum_{j=0}^{N-1}|z_j|^2\right], \tag{1}$$

where Δ_N is the Vandermonde determinant

$$\Delta_N=\prod_{0\le j<k\le N-1}(z_j-z_k) \tag{2}$$

and $m=2p+1$ is an *odd* integer [distances are measured in terms of the magnetic length $(\hbar c/eB)^{1/2}$]. The wave function $|m\rangle$ is antisymmetric in each pair of coordinates. When one z is rotated about another by θ, the wave function changes by $\exp(im\theta)$, and so there are m flux quanta per electron. If all of the z are rotated by $\exp(i\theta)$, the wave function changes by $\exp[iN(N-1)m\theta/2]$.

To connect Laughlin's picture to the theory of random matrices, consider the electrons as objects of indefinite position described by a coordinate matrix Q_{jk} of rank N whose eigenvalues are the complex coordinates z_j. (Since Q has complex eigenvalues, it is, in particular, not Hermitian). Choose the matrices Q according to a probability distribution $P(Q)\mu(dQ)$, where $\mu(dQ)$ is the linear measure

$$\mu(dQ)=\prod_{0\le j,k\le N-1}dQ_{jk}\,dQ_{jk}^* \tag{3a}$$

and

$$P(Q)\equiv\exp[-S_0(Q)]\ , \tag{3b}$$

$$S_0(Q)\equiv\frac{1}{2}\operatorname{Tr}(Q^\dagger Q)\ . \tag{3c}$$

Thus Q varies over all complex matrices C. Except in regions of lower dimensionality which are irrelevant to the probability distribution, the eigenvalues are distinct. Define the matrix $E=\operatorname{diag}(z_0,\ldots,z_{N-1})$ and let X be the $N\times N$ matrix whose columns are the eigenvectors of Q (i.e., essentially the electron wave functions). Then X is nonsingular and $Q=XEX^{-1}$. The distribution P_c of eigenvalues of Q is then obtained by integrating over all complex matrices X, and is given by[7,5]

$$P_c(z_0,\ldots,z_{N-1})=K_c|\Delta_N|^2\exp\left[-\frac{1}{2}\sum_{l=0}^{N-1}|z_l|^2\right],\qquad K_c^{-1}=(2\pi)^N\prod_{j=0}^{N-1}[(j+1)!]\ . \tag{4}$$

Of course, $P_c=\langle 1|1\rangle$ is just the square of the $m=1$ Laughlin wave function. Note in particular that the Fermi statistics of electrons are automatically accounted for by the well-known "repulsion" of eigenvalues of a random matrix.[5]

More is needed, however, to complete the full Laughlin picture for general m. The major obstacle is the above-mentioned eigenvalue repulsion, which has the practical consequence in the random-matrix model Eq. (3) that powers of z_j greater than $N-1$ are not allowed. In other words, each coordinate taken separately has at most angular momentum $N-1$. It is necessary to add up to an additional $2p(N-1)$ units of angular momentum to each of the fermionic degrees of freedom in the system

$[2p+1=m$, as in Eq. (1)].

Define a set of matrices $F_{I,J}$, $I,J=0,1,\ldots,N-1$

$$F_{I,J}=\mathrm{Tr}(Q^{I+J}) \tag{5a}$$

and their Fourier transform

$$\tilde{F}_{K,L}=\frac{1}{N}\sum_{I,J=0}^{N-1}\exp[2\pi i(IK+JL)/N]F_{I,J} \ . \tag{5b}$$

Both F and $\tilde{F}$ depend only on the eigenvalues of Q, and are totally symmetric in them. Thus under a rotation $z \to \exp(i\theta)z$ by $\theta=2\pi/N$,

$$F_{I,J}\to\exp[2\pi i(I+J)/N]F_{I,J} \ , \tag{6a}$$

$$\tilde{F}_{K,L}\to\tilde{F}_{K+1,L+1} \ . \tag{6b}$$

Thus if a term

$$S_F(p)=\sum_{0\le I,J,K\le N-1}\bar{\psi}_I(\widetilde{F^p})^{\dagger}_{IJ}(\widetilde{F^p})_{JK}\psi_K \tag{7}$$

is added to the action S_0, a rotation by $2\pi/Np$ reduces to a cyclic permutation of the $\{\bar{\psi}\}$ and $\{\psi\}$, thus accounting for the Pauli principle. These Grassmann variables are taken as *spinless* as it is assumed that the magnetic field freezes out the spin degrees of freedom. The interaction term Eq. (7) thus couples up to $\max[(I+J)p]=2p(N-1)$ units of angular momentum to fermionic degrees of freedom, as required.

When the integration over the $2N$ Grassmann coordinates $\{\bar{\psi},\psi\}$ is performed, the result is

$$\int D\bar{\psi}D\psi\exp[-S_F(p)]=|\mathrm{Det}\tilde{F}|^{2p} \ . \tag{8}$$

The right-hand side of Eq. (8) can be expressed simply as follows:

$$\begin{aligned}|\mathrm{Det}\tilde{F}|&=|\mathrm{Det}F|\\&=|\mathrm{Det}_{IJ}\,\mathrm{Tr}(Q^{I+J})|\\&=|\mathrm{Det}_{IJ}\mathrm{Tr}(E^{I+J})|\\&=\left|\mathrm{Det}_{IJ}\sum_{j=0}^{N-1}z_j^{I+J}\right|\\&=|(\mathrm{Det}_{iI}z_i^I)(\mathrm{Det}_{jJ}z_j^J)|\\&=|(\Delta_N)^2| \ .\end{aligned} \tag{9}$$

Thus the eigenvalues of complex matrices Q chosen with weight $\exp[-S_0-S_F(p)]\,D\bar{\psi}D\psi\mu(dQ)$ are distributed according to

$$|\Delta_N|^{4p+2}\exp\left[-\tfrac{1}{2}\sum_{j=0}^{N-1}|z_j|^2\right]=\langle 2p+1|2p+1\rangle \ , \tag{10}$$

which is perforce the absolute square of the Laughlin wave function for $m=2p+1$. Note that the construction Eq. (6) with spinless fermions ensures that $m=2p+1$ is an odd integer. Had fermions of spin s been used, the result would have been $m=2p(2s+1)+1$. *The allowed values of m thus depend upon the number of the fermionic degrees of freedom to which the magnetic field couples.* This restriction may have implications for the allowed values of m in the FQHE.

It is interesting to interpret the above discussion in terms of random-matrix surface models. The fermionic action $S_F(p)$ generates diagrams with vertices which have up to $2p(N-1)$ legs. Each leg corresponds to a unit of angular momentum. Thus in some sense the random-matrix approach is equivalent to counting the number of ways a surface can be covered by various polygons, with each covering weighted by the Pauli principle manifest in the Grassmann algebra. This might be a reflection of the underlying "string-theoretic" nature of the interaction of magnetic vortices and electrons.

In order to complete the analogy with the Laughlin model, the effects of quasihole excitations are now considered. In the original droplet wave functions $|m\rangle$, there are m flux quanta per electron. Quasiholes are two-dimensional bubbles in the fluid which act like flux tubes with a single unit of flux. Each quasihole removes $1/m$ of an electron charge, and thus[8] displays "fractional statistics."

The effect of a quasihole excitation at complex coordinate η can be included in $S_F(p)$ by replacing the $\{\psi\}$ in Eq. (7) with

$$\psi_I\to\sum_{J=0}^{N-1}(\eta\delta_{IJ}-Q_{IJ})\psi_J \tag{11}$$

and similarly for the $\{\bar{\psi}\}$. This leads to an additional factor of

$$\left|\mathrm{Det}(\eta\cdot 1-Q)\right|^2=\left|\prod_{j=0}^{N-1}(\eta-z_j)\right|^2 \tag{12}$$

in the overall probability distribution, just as in the Laughlin model. Of course, it is possible to continue the hierarchy[9] by taking the η to be dynamic variables equal to the eigenvalues of a new matrix.

In conclusion, therefore, a random-matrix version of the Laughlin model of the FQHE has been formulated, thus demonstrating that fractional statistics can occur in random-matrix models. Such a formulation may lead to new insights and connections with existing models, as well as the possibility of speeding direct computations. For instance, the random-matrix formulation is very similar to a lattice field theory, for which numerical techniques are readily available.[10]

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