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IS THE STANDARD MODEL HIGGS MASS PREDICTABLE?

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Strong evidence supports the idea that pure ϕ^4 field theory is "trivial" or non-interacting. Should such a situation persist when gauge fields are present, it is fair to question the idea of symmetry breaking (and gauge boson mass generation) by elementary scalars. An alternative possibility, suggested by the consideration of toy models, is that a consistent non-trivial theory may exist for only a select range of parameters, implying for example that the standard model Higgs mass may be bounded or calculable. In principle the Monte Carlo renormalization group can be used to examine this possibility. Here the formalism is elucidated and the results of the first calculation for the standard model are presented.

1. Prolegomena

Over two decades ago it was suggested [1] that gauge particles could be rendered massive in a theory if elementary scalar particles exist. These scalars supposedly interact with themselves in such a fashion [2] as to generate a vacuum state which lacks a symmetry of the action. Gauge particles propagating in the asymmetric vacuum appear to be massive, while the underlying theory remains renormalizable. This idea of "spontaneous" symmetry breaking in fact is an integral part of the standard model [3] of the weak interaction.

Nevertheless a compelling variety of evidence [4–6] suggests that pure ϕ^4 theory is "trivial" or non-interacting in four dimensions, i.e. that in this limit no spontaneous symmetry breaking occurs. In certain cases (such as the large-N limit of the $O(N)$ gauge model) gauge fields seem to "rescue" a ϕ^4 theory from triviality provided that the parameters of the renormalized theory obey certain restrictions [7]. The situation with the standard model remains an open question, although non-perturbative methods have been applied [7–10] to address this question in

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simpler gauge-Higgs systems. Here we present results from a first non-perturbative study of the possible triviality of the standard model.

It should be mentioned that (in contrast to a popular misconception) it is not in general necessary for a theory to be asymptotically free in order for it to be non-trivial. It is only necessary that the running coupling constants approach fixed-point values (corresponding to zeros of the beta functions) as the momentum scale increases. A fixed-point value can even be infinite, corresponding to an appropriately-defined zero of the beta function at infinite coupling. What must be avoided is a "Landau ghost", where the running coupling becomes infinite at a finite momentum scale. Thus asymptotic freedom is *sufficient* to guarantee non-triviality (an asymptotically-free theory has an attractive fixed point at zero coupling), but *unnecessary*. Whether or not the theory is asymptotically free, the domain of attraction (i.e. the set of values of the renormalized coupling constants for which the running couplings approach the fixed point) of the fixed point determines the allowed range of parameters for a non-trivial theory. If this domain of attraction is finite, an (upper) bound on the Higgs mass is a likely consequence [11–13]; if it is limited to a single point, the Higgs mass is predictable [11, 7, 14]. The implications of these ideas for real-space renormalization have been discussed by Hasenfratz [4] and ourselves [6, 7, 9] and are reviewed below.

The organization of the paper is as follows: first, a brief review of the ideas of the real-space renormalization group is given, followed by a definition of the $SU(2) \times U(1)_Y$ lattice Higgs model. Various limit models are discussed, and finally the full $SU(2) \times U(1)_Y$ model is studied. The results are discussed in light of the above remarks.

2. Triviality pursuit: A description of the procedure

How can the question of the triviality or non-triviality of a theory be answered in a sensible fashion? One way of approaching these questions is via the Monte Carlo renormalization group (MCRG). By dealing directly with the scaling properties of the various field and operators under consideration, it allows an economical (albeit numerical) formulation of the problem. Additionally, by restricting the analysis to a few carefully chosen operators, the MCRG allows one to introduce accurate and systematically improvable approximations. More direct methods [5] of studying triviality do not allow this simplification.

The MCRG [15–17, 6, 7] is a combination of the ideas of Monte Carlo simulation [18] with those of the real space renormalization group [19]. The system under consideration is divided up into "blocks" or collections of site variables, and a smaller set of block variables is defined by averaging in some fashion over these site variables. The new block system is governed by a block-renormalized action (defined by couplings $\{K'\}$) in the same fashion as the original site action (defined by couplings $\{K\}$) governed the site system. By studying the "flow" of couplings

$\{K\} \to \{K'\} \to \{K''\} \to \cdots$ as repeated blockings are performed, the critical properties of the system can be determined. For a lattice gauge theory, these critical properties determine the continuum limit of the theory, since in this limit the lattice spacing in units of the correlation length must vanish, implying a divergent correlation length. Any continuum limit therefore lies on a critical surface. The lore of the renormalization group [19] tells us that the critical behaviour at all of the uncountably infinite number of points on this surface can be ascertained by the consideration of the renormalization flow in the neighbourhood of the fixed points which control it. The efficiency of such a reduction is difficult to miss.

A second important point to keep in mind is the fact that the number of *relevant* parameters for the continuum theories associated with a given fixed point is equal to the number of its independent renormalized couplings. Here, the word "relevant" is used in the sense of the renormalization group, i.e. it refers to the number of directions in which the renormalization group trajectories flow away from the fixed point. Thus if the number of relevant directions for a fixed point is less than the number of renormalized coupling constants, one or more of these couplings (e.g. the Higgs mass) can be calculated in terms of the others. This quantum "parameter reduction" [11] is perhaps easiest to see within the framework of the renormalization group formalism.

These ideas can be formulated in a more precise mathematical fashion as follows. Consider a system of site variables $\{\phi\}$ governed by a site action $S\{\phi\}$. A block-renormalized action $S'\{\phi'\}$ is determined from the site action by use of the projection operator $P[\{\phi'\};\{\phi\}]$:

$$\exp[-S'\{\phi'\}] = \mathrm{Tr}_{\{\phi\}} P[\{\phi'\};\{\phi\}]\exp[-S(\phi)], \tag{1}$$

where the trace (or functional integral) is only over the site variables $\{\phi\}$. The requirement that the renormalization group transformation preserve the partition function (vacuum-to-vacuum transition amplitude) of the system imposes the constraint

$$\mathrm{Tr}_{\{\phi'\}} P[\{\phi'\};\{\phi\}] = 1. \tag{2}$$

The projection operator is otherwise entirely arbitrary.

In order to extract the flow diagram of a theory via the MCRG, it is necessary to calculate the block action. General methods for determining the block action have been given elsewhere [16, 17, 6, 7, 21]. The specific method [6, 21] used here is based upon the following observation [6]: suppose that one had a set of values $x_1, x_2 \ldots$ of a random variable x distributed according to a probability measure

$$P(x)\,\mathrm{d}x = N(\lambda)\,\mathrm{e}^{-\lambda x^4}\mathrm{d}x \tag{3}$$

over the interval $-\infty < x < +\infty$ with λ an unknown parameter to be determined.

Then, since

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}\left[x^n \mathrm{e}^{-\lambda x^4}\right] = 0 \tag{4a}$$

it follows that

$$n\langle x^{n-1}\rangle - 4\lambda\langle x^{3+n}\rangle = 0, \tag{4b}$$

and thus λ can be determined by measuring moments of x over the distribution. This trick [6] is a consequence [21] of the invariance of the integration measure $\mathrm{d}x$ and domain $(-\infty, \infty)$ under infinitesimal translations $x \to x' = x + \varepsilon$:

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x f(x) &= \int_{-\infty}^{+\infty} \mathrm{d}x' f(x') \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x f(x) - \varepsilon \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \cdots,\end{aligned} \tag{5a}$$

so that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = 0. \tag{5b}$$

Note that $f(x)$ cannot be invariant under the reflection $x \to -x$ (e.g., n must be odd in eq. (46)), else eq. (5b) is only a trivial $(0 = 0)$ identity.

When this concept is applied to path integrals [6, 21, 9, 10], the result is

$$\int \mathscr{D}\phi' \frac{\delta}{\delta\phi'(x)}\left[O_n\{\phi'\}\exp(-S\{\phi'\})\right] = 0, \tag{6}$$

provided that the quantity in square braces [...] vanishes or is periodic at the limits of integration. If a set of functionals $O_n\{\phi'\}$ is inserted in eq. (6), the result is a system of simultaneous equations for the couplings $\{K'\}$ which determine the block action $S'\{\phi'\}$ for the theory. Thus the block renormalized action can be readily determined by measuring the expectation values of various functionals over the space of block fields. Applications of this technique to multicomponent systems are discussed in sect. 4.

3. The $\mathrm{SU(2)} \times \mathrm{U(1)}_Y$ lattice theory

In the lattice standard model considered here, the fermionic sector of the theory is neglected. This simplification is essentially forced upon us, for the inclusion of chiral fermions in a lattice theory is an extraordinarily difficult problem [22]. However, if only light fermions are considered, the associated Yukawa couplings are quite small (e.g. $\sim 10^{-5}$ for the electron) and are therefore likely negligible. Heavy fermions (i.e. those whose masses are an appreciable fraction of the ~ 200 GeV mass scale of the weak interaction) may require further consideration [12].

The bosonic sector of the lattice standard model (employing the usual [23] formalism for lattice gauge fields) is defined by the partition function

$$_{\infty}\langle 0|0\rangle_{-\infty} \equiv Z \equiv \int [\prod \mathrm{d}U][\prod \mathrm{d}V][\prod \mathrm{d}\Phi^{+}][\prod \mathrm{d}\Phi]\mathrm{e}^{-S}, \tag{7}$$

where

$$\begin{aligned} S &= S_1 + S_2 + S_{\mathrm{V}} + S_{\mathrm{H}}, \\ S_1 &= \beta_1 \sum_{\square} [1 - \mathrm{Re}\, U_{\square}], \\ S_2 &= \beta_2 \sum_{\square} [1 - \tfrac{1}{2}\mathrm{Tr}\, V_{\square}], \\ S_{\mathrm{V}} &= -2 \sum_{n,\mu} \mathrm{Re}\{ \phi_n^{+} U_{n,\mu} V_{n,\mu} \phi_{n+\mu} \}, \\ S_{\mathrm{H}} &= \lambda \sum_{n} (\Phi_n^{+}\phi_n - k)^2. \end{aligned} \tag{8}$$

Here U and V denote respectively the U(1) and SU(2) plaquette products, and

$$\Phi = \begin{pmatrix} \phi_{+} \\ \phi_0 \end{pmatrix} \tag{9}$$

is the (complex, $I_{\mathrm{W}} = \frac{1}{2}$, $Y = 1$) Higgs field associated with each site n. Links $U_{n,\mu}$ and $V_{n,\mu}$ connect sites n with nearest neighbour sites in direction μ, and each term in each sum of eqs. (8) appears exactly once. The model defined by eqs. (8) has the correct formal continuum limit of the standard model [24].

It is useful at this point to introduce the simplification of a fixed-length scalar field. This reduction is equivalent to taking the limit of large λ in eqs. (8), the consequences of which are discussed in sect. 5. Following the rescaling $\Phi \to \sqrt{\kappa}\,\Phi$, the action is given by

$$S = S_1 + S_2 + S_{\mathrm{L}}, \tag{10a}$$

where

$$\begin{aligned} S_{\mathrm{L}} &= -2\kappa \sum_{n,\mu} \mathrm{Re}\{ \Phi_n^{+} U_{n,\mu} V_{n,\mu} \Phi_{n+\mu} \}, \\ &\Phi_n^{+} \cdot \Phi_n = 1, \qquad \forall n. \end{aligned} \tag{10b}$$

This theory possesses [24] three phases:

(i) a *confined* phase, in which the free energy required to separate a pair of test charges increases without bound as the distance between them grows;

(ii) an *electrodynamics* or "Coulomb" phase, where interparticle forces between test charges obey Coulomb's law; and

(iii) a "*Higgs*" phase, where spontaneous symmetry breakdown occurs. The significance of these phases for the flow diagram is discussed below.

4. Limits of the lattice standard model

It is worthwhile to examine several limiting cases of the $SU(2)\times U(1)$ Higgs theory before presenting the results for the full model. Specifically, we consider (i) the O(4) XY-model (or non-linear sigma model), which is the limit of large β_1 and β_2; (ii) the U(1)-Higgs limit of large β_2 (note that the Higgs field transforms as a global SU(2) doublet); and finally (iii) the SU(2)-Higgs limit of large β_1.

4.1. THE O(4) XY MODEL (NON-LINEAR SIGMA MODEL)

This model is the limit of the fixed-length $SU(2)\times U(1)$ lattice standard model when β_1 and β_2 increase without bound. It is defined by an action

$$S_{xy} = -2\kappa \sum_{n,\mu} F_n \cdot F_{n+\hat{\mu}}\,, \tag{11a}$$

where n is a lattice site and μ is one of the four unit vectors connecting that site to its nearest neighbour in one direction. Thus each nearest-neighbour pair is summed *once*. The F_n are column vectors of four real fields $f_{n,i}$:

$$F_n = \begin{pmatrix} f_{n,1} \\ f_{n,2} \\ f_{n,3} \\ f_{n,4} \end{pmatrix}, \tag{11b}$$

subject to the constraint

$$\sum_i (f_{n,i})^2 = 1\,. \tag{11c}$$

For reasons which will become evident in the SU(2)-Higgs limit discussed in subsect. 4.3, we have chosen a scale factor of 3 for the blocking transformation. Thus the original (or "site") system of size L^d is divided into $(L/3)^d$ hypercubes, each containing 3^d sites. The block fields $\{F'\}$ are then calculated by averaging the $\{F\}$ over each block. Specifically,

$$P[\{F',F\}]\mathscr{D}F = \prod_{\{f'\}} \delta\left[f'_{n',i} - N_{n'}^{-1} \sum_{n\in \text{block}} f_{n,i}\right] \prod_{\{f\}} \mathrm{d}f_n\,, \tag{12a}$$

where

$$N_{n'} \equiv \left[\sum_i \Big(\sum_{n \in \text{block}} f_{n,i} \Big)^2 \right]^{1/2} \tag{12b}$$

ensures that $F'_{n'} \cdot F'_{n'} = 1$.

We find that the block action is well-described by a form

$$S'_{xy} \simeq -2\kappa' \sum_{n', \mu'} F'_{n'} \cdot F'_{n'+\mu'} . \tag{13}$$

(Of course, in general S' contains an arbitrary number of terms. However, if the blocking transformation is sufficiently comprehensive, the assumption eq. (13) is a highly accurate truncation, and leads to the definition of an "effective" coupling κ'_{eff} which describes well the critical behaviour of the system.)

The (effective) block coupling κ' can be derived by a generalization of the method described in sect. 2. Rather than varying a single field, however, an infinitesimal rotation in group space is performed. Under such a rotation,

$$\begin{aligned} F'_{n'} &\to \left(1 + \varepsilon_{ij} R_{ij}\right) F'_{n'} \\ &\equiv F'_{n'} + \Delta_{n'} F'_{n'} , \end{aligned} \tag{14a}$$

where R_{ij} is the rotation matrix

$$\left(R_{ij}\right)_{ab} \equiv \varepsilon_{ijab} . \tag{14b}$$

In eq. (6) the fact that the measure $\mathscr{D}\phi'$ is invariant under infinitesimal translations leads to a set of identities between various expectation values. Here, the integration measure $\mathscr{D}F'$ used in the definition of the vacuum-to-vacuum transition amplitude (the "partition function") Z

$${}_{+\infty}\langle 0_{xy} | 0_{xy} \rangle_{-\infty} \equiv Z_{xy} \equiv \int \mathscr{D}F' \, \mathrm{e}^{-S'_{xy}} \tag{15a}$$

is invariant under group *rotations* eq. (14) and leads to a corresponding set of identities (cf. eq. (5b)):

$$\int \mathscr{D}F' \Delta_{n'} \left[O_{n'}\{F'\} \mathrm{e}^{-S'_{xy}} \right] = 0 . \tag{15b}$$

In eqs. (5), $f(x)$ could not be invariant under the reflection $x \to -x$, else only a trivial identity would be implied by eq. (55b). The corresponding requirement here is that $O_{n'}\{F'\}$ not be rotationally invariant. Instead each $O_{n'}\{F'\}$ must be of the form

$$O_{n'}\{F'\} = \Delta_{n'} I\{F'\}, \tag{16}$$

where $I\{F'\}$ is invariant under rotations (i.e. is constructed from quantities like $F'_{a'} \cdot F'_{b'}$). It is natural to choose

$$I\{F'\} = S'_{xy}\{F'\}, \tag{17}$$

which leads to the equations

$$\langle F'_{n'} R_{ij} R_{kl} G'_{n'} \rangle + 2\kappa' \big\langle [F'_{n'} R_{ij} G'_{n'}][F'_{n'} R_{kl} G'_{n'}] \big\rangle = 0, \tag{18a}$$

where

$$G'_{n'} \equiv \sum_{\mu} (F'_{n'+\mu} + F'_{n'-\mu}). \tag{18b}$$

Equations (18) are a trivial identity unless i and j equal k and l in either order. Contraction with $\delta_{ik}\delta_{jl}$ with the use of the identities

$$\varepsilon_{ijab}\varepsilon_{ijcd} = 2(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}),$$

$$R_{ij} R_{ij} = -6 \cdot \mathbf{1}, \tag{19}$$

yields the result

$$2\kappa' = \frac{\Sigma_{n'} 3 \langle F'_{n'} \cdot G'_{n'} \rangle}{\Sigma_{n'} \langle (G'_{n'} \cdot G'_{n'}) - (G'_{n'} \cdot F'_{n'})^2 \rangle} \tag{20a}$$

$$= \frac{\Sigma_{n'} \langle G_{\parallel, n'} \rangle}{\Sigma_{n'} \frac{1}{3} \langle (G_{\perp, n'})^2 \rangle}, \tag{20b}$$

where in an obvious notation

$$G'_{\parallel} \equiv F' \cdot G', \tag{20c}$$

$$G'_{\perp} \equiv G' - (F' \cdot G') F', \tag{20d}$$

so that

$$F' \cdot G'_{\perp} = 0, \tag{20e}$$

$$G'^2 = G'^2_{\parallel} + G'^2_{\perp}. \tag{20f}$$

(The extra sum on n' is introduced in eqs. (20) to restore translation invariance.) Of course, many other functions $I\{F'\}$ can be used in eq. (16) to generate different

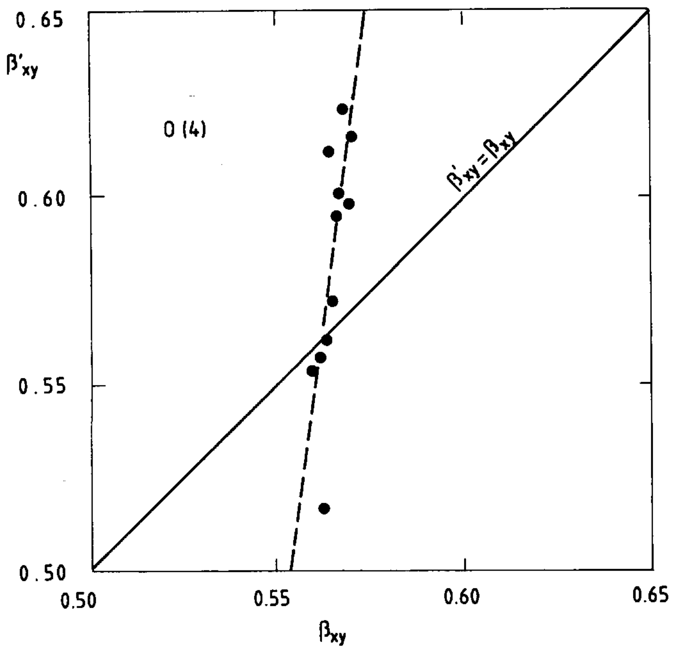

Fig. 1. Plot of κ' versus κ for the four-dimensional O(4) XY model. The solid line ($\kappa' = \kappa$) indicates the location of the fixed point κ^*; the broken line is the least-squares fit.

equations and to extract any other couplings present in the block action. Perhaps surprisingly however, the simple truncation eq. (13) provides a reasonable description of the critical behaviour of this model.

This O(4) model was simulated on a 6^4 lattice, using the standard algorithm of Metropolis et al. [25]. The block coupling κ' (as defined by eq. (20)) was measured over 1500 iterations, following 150 "equilibration" iterations. The result is shown in fig. 1. A fixed point is clearly in evidence, and a least-squares fit to the 11 points shown yields

$$2\kappa^* \equiv \beta^*_{xy} = 0.56\,, \tag{21a}$$

$$3^y \equiv \left.\frac{\partial \kappa'}{\partial \kappa}\right|_{\kappa'} = 7.8\,, \tag{21b}$$

$$y = 1.9\,, \tag{21c}$$

$$\nu = y^{-1} = 0.54\,, \tag{21d}$$

where the 3 in eq. (21b) is just the scale factor of the blocking transformation.

These figures can be compared with known results. A rigorous lower bound on the critical coupling is provided [25] by the mean field theory estimate:

$$2\kappa^{\mathrm{c}} = \beta_{xy}^{\mathrm{c}} \geqslant \tfrac{1}{2}, \tag{22}$$

while a rigorous upper bound is given by [26, 24]

$$2\kappa^{\mathrm{c}} = \beta_{xy}^{\mathrm{c}} \leqslant 0.622. \tag{23}$$

Additionally, an expansion in powers of the inverse coordination number q^{-1} (for a hypercubic lattice $q = 8$) yields [27, 24]

$$2\kappa^{\mathrm{c}} = \beta_{xy}^{\mathrm{c}} = 0.6055 + \mathrm{O}(q^{-6}), \tag{24}$$

where the q^{-5} term makes a fractional contribution of 5×10^{-3}.

For the exponent ν, the mean-field exponent $\nu = \frac{1}{2}$ is expected. The anomalous dimension of the field F^2 is given by [28]

$$\gamma_{F^2} = 2 - \nu^{-1} \tag{25}$$

and vanishes if $\nu = \frac{1}{2}$. It can be seen that the results of our truncated calculation agree with known values. Especially interesting is the fact that κ^* well approximates κ^{c}, for this close correspondence is *not* guaranteed by universality, and thus provides a stringent check of eq. (13). We therefore proceed with confidence to the next limit.

4.2. THE U(1)-HIGGS LIMIT MODEL

In the limit of large β_2, the system reduces to a U(1) gauge field coupled to a fixed-length scalar with a global SU(2) symmetry. This model has been examined previously [24, 29] and exhibits a distinct three-phase structure. The simpler abelian Higgs model with a single (complex) component scalar field has also been well-studied by analytical [30] and numerical [31, 10] techniques.

This U(1)-Higgs limit model is defined by an action

$$\begin{aligned} S_{\mathrm{U1}} &= S_1 + S_{\mathrm{L1}}, \\ S_1 &= \beta_1 \sum_{\square} (1 - \cos\theta_{\square}), \\ S_{\mathrm{L1}} &= -2\kappa \sum_{n,\mu} \mathrm{Re}\left(\Phi_n^+ U_{n,\mu} \Phi_{n+\hat{\mu}}\right), \end{aligned} \tag{26a}$$

with

$$\begin{aligned} \theta_{\square} &\approx \theta_{n,\mu} + \theta_{n+\mu,\nu} - \theta_{n+\nu,\mu} - \theta_{n,\nu}, \\ U_{n,\mu} &\equiv \exp(i\theta_{n,\mu}), \end{aligned} \tag{26b}$$

where Φ is the complex two-component scalar field:

$$\Phi \equiv \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix},$$

$$|\phi_+|^2 + |\phi_-|^2 = 1. \tag{26c}$$

The connection with the O(4) XY model (non-linear sigma model) discussed above is more transparent when S_{L1} is rewritten in terms of the real four-component fields F_n of eq. (11b) and matrices R_{ij} of eq. (14b):

$$S_{L1} = -2\kappa \sum_{n,\mu} F_n \exp\left[-\theta_{n,\mu}(R_{12} + R_{34})\right] F_{n+\mu}. \tag{26d}$$

The blocking transformation for the scalar field is a gauge-invariant generalization of eqs. (12). The block scalar field $\Phi'_{n'}$ is defined by the projection operator

$$P[\{\Phi'\},\{\Phi\}]\mathscr{D}\Phi = \prod_{\{\Phi'\}} \delta\left[\Phi'_{n'} - \tilde{\Phi}_{n'}\{\Phi\}\right]\mathscr{D}\Phi, \tag{27a}$$

where

$$\tilde{\Phi}_{n'}\{\Phi\} \equiv \frac{1}{N_{n'}} \sum_{n \in \text{block}} U_{n,\mu} U_{n+\mu,\nu} \dots \Phi_n. \tag{27b}$$

In other words, the block scalar field Φ'_n is simply the average of the (parallel-transported) site scalar fields Φ_n in the block. The factor $N_{n'}$ enforces the constraint

$$|\Phi'_{n'}|^2 = 1. \tag{27c}$$

The block point is defined to be that site point in the cube which is nearest to the origin (all coordinates are taken as positive in the periodic lattice). Typically multiple paths of parallel transport exist from this block point n' to each site point n in eqs. (27). In such cases, the arithmetic average of all of the *shortest* paths is used, as in ref. [10].

The projection operator for the gauge fields $\{\theta'\}$ is defined by:

$$P[\{\theta'\},\{\theta\}]\mathscr{D}\theta \equiv \prod_{ij} \delta\left[\theta'_{ij} - \tilde{\theta}_{ij}\{\theta\}\right]\mathscr{D}\theta, \tag{28a}$$

where the $\tilde{\theta}_{ij}$ are the arguments of the paths shown in fig. 2:

$$\tilde{\theta}_{ij} \equiv \arg\left\{ \sum_{\text{paths}} U_{ia} U_{ab} \dots U_{bj} \right\}. \tag{28b}$$

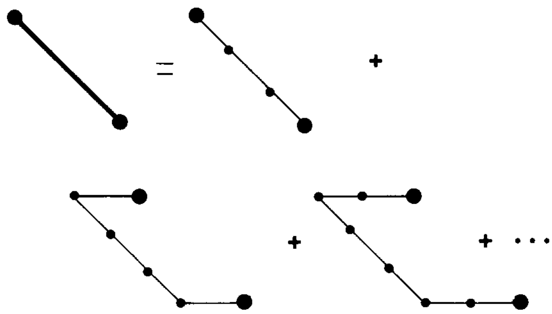

Fig. 2. Schema of paths included in the U(1) block link. The ellipsis (...) refers to obvious rotations.

The inclusion of these paths allows a good description of the critical behaviour expected [32–34] at $\beta \sim 1$.

Next the method of extracting the block action is described. The block action is approximated by a form

$$S'_{\mathrm{U1}} \cong \beta'_1 \sum_{\square} [1 - \cos\theta'_{\square}] - 2\kappa' \sum_{n,\mu} \mathrm{Re}\left(\Phi_n'^{+} U_{n,\mu} \Phi'_{n+\hat{\mu}}\right) \tag{28c}$$

(cf. eqs. (26)). If the procedure used to obtain eq. (20) is applied to this U(1)-Higgs limit model, the result is

$$2\kappa' = \frac{\Sigma_n 3\langle \mathrm{Re}(c_n'^{+} \cdot \Phi'_N)\rangle}{\Sigma_n \left\langle c_n'^{+} \cdot c_n - \left[\mathrm{Re}(c_n'^{+} \cdot \Phi'_n)\right]^2 \right\rangle} = \frac{\Sigma_n \langle c'_{\|,n}\rangle}{\Sigma_n \frac{1}{3}\langle |c'_{\perp,n}|^2\rangle}, \tag{29a}$$

where

$$c'_n \equiv \sum_{\mu} \left(U'_{n,\mu}\Phi'_{n+\mu} + U'^{*}_{n-\mu,\mu}\Phi'_{n-\mu}\right) \tag{29b}$$

and

$$c'_{\|} \equiv \mathrm{Re}(c'^{+} \cdot \Phi'),$$

$$c'_{\perp} \equiv c' - \mathrm{Re}(c'^{+} \cdot \Phi')\Phi' \tag{29c}$$

so that

$$\mathrm{Re}(c_{\perp}'^{+} \cdot \Phi') = 0, \tag{29d}$$

$$|c'|^2 = c_{\|}'^2 + |c'_{\perp}|^2. \tag{29e}$$

The equation for β_1' follows from (cf. eq. (15)):

$$\int \mathscr{D}\theta' \frac{\delta}{\delta\theta_{ij}} \left[O\{\theta'\} e^{-S'_{U1}} \right] = 0 . \tag{30}$$

Equation (30) (like eq. (5b)) is a trivial $(0 = 0)$ identity unless $O\{\theta'\}$ is odd in the $\{\theta'\}$. It is convenient to choose

$$O\{\theta'\} = \frac{\delta S_1'}{\delta\theta_{ij}'} \cdot \frac{1}{\beta_1'} \equiv \frac{\delta}{\delta\theta_{ij}'} \sum_{\square} (1 - \cos\theta_{\square}') \tag{31}$$

(compare eqs. (16) and (17)). Equations (30) and (31) lead to a simple linear equation for β_1' in terms of κ' and various expectation values. (If, instead of eq. (31), $O\{\theta'\} = \delta S'/\delta\theta_{ij}'$ had been employed, a *quadratic* equation for β_1' would have resulted.) A similar procedure was employed by us [9,10] to study the abelian Higgs model. Note that in the event that corrections to eq. (28) are required, other operators $O_1, O_2, O_3, \ldots$ can be inserted in eq. (30) in order to generate a set of equations for all couplings in the theory. Thus our calculation can be systematically improved to arbitrary accuracy.

The numerical results are presented below. In all cases the theory was formulated upon a 6^4 lattice and blocked to a 2^4 lattice. The standard routine of Metropolis et al. [25] was employed for both the gauge and scalar fields. The step size $\Delta\theta$ for

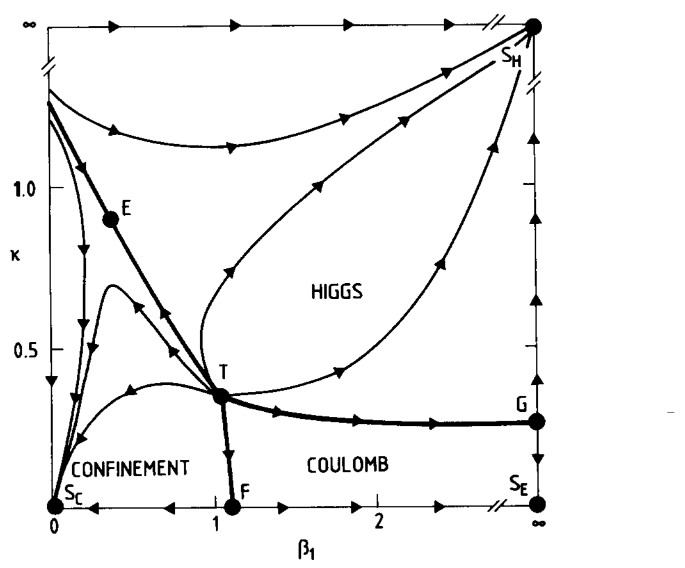

Fig. 3. Flow diagram for U(1) gauge theory coupled to a (globally) SU(2)-invariant scalar field.

the gauge field was readjusted after each update of the lattice to give approximately 50% acceptance. New "trial" scalar fields Φ were generated randomly on a four-sphere, with up to 20 hits per site applied to ensure greater than 90% overall acceptance.

The flow diagram for this U(1)-Higgs limit model is shown in fig. 3. Approximately 100 points were used to generate this flow diagram. Four fixed points are present, labelled E, F, G, and T. The flow diagram corresponds well with the phase diagram [24, 29] of the theory. This close correspondence is an indication that our truncation approximation (eq. (28)) likely provides a valid description of the critical behaviour of the model. Each of the three phases (confinement, Higgs, and Coulomb or "electrodynamics") is distinct and possesses its own sink (labelled respectively S_C, S_H, or S_E).

The fixed point G occurs in the limit of large β_1, where the theory approaches the O(4) XY model (or non-linear sigma model) discussed in subsect. 4.1. The critical values for this fixed point have already been given in eqs. (21), and are consistent with the numbers expected for a trivial gaussian theory. Much more interesting is the fact that the gauge coupling $g_1^2 = \beta_1^{-1}$ is irrelevant along the Higgs-Coulomb phase transition. Thus (except for possibly the tricritical point T) in this scenario the entire Higgs-Coulomb phase transition has mean-field gaussian critical exponents.

Consider next the pure U(1) gauge limit ($\kappa = 0$). A fixed point is clearly in evidence (see fig. 4) in a plot of β' versus β; however, the critical region is quite narrow and we expect relatively large errors in the determination of the associated critical exponent. Nevertheless a least-squares fit to the five points between $\beta = 1.054$

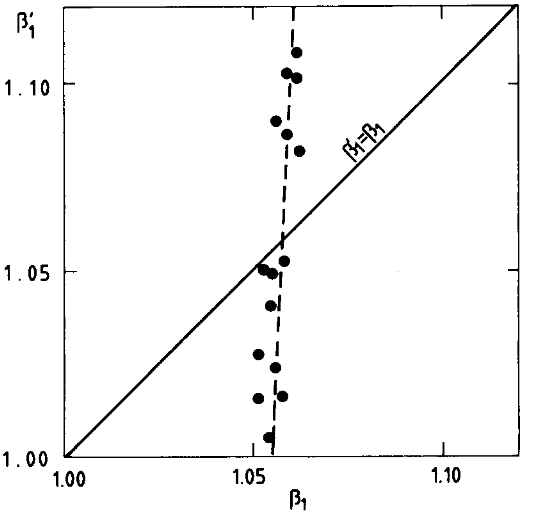

Fig. 4. Plot of β' versus β for a pure U(1) gauge theory. The solid line ($\beta' = \beta$) indicates the location of the fixed point β^*; the broken line displays the result of a least squares fit.

and $\beta = 1.056$ yields

$$\beta^* = 1.055\,, \tag{32a}$$

$$3^{y_F} \equiv \frac{\partial\beta'}{\partial\beta} = 25.0\,, \tag{32b}$$

$$y_F = 2.9 \pm 1.0\,. \tag{32c}$$

Each point in fig. 4 was generated by measuring β' over 1250 iterations following thermalization for 125 iterations.

These critical indices may be compared with the result $y = 4$ (corresponding to $\partial\beta'/\partial\beta = 81$) expected [35] for a first-order phase transition. We found some evidence suggestive of a first-order phase transition in the compact U(1) theory [10] and in the large-κ limit of the non-compact U(1) theory [9] (dual to the compact model in this limit [36]). However, a demonstration of a first order transition is in principle impossible within the MCRG, for y must equal four (i.e. the number of space-time dimensions) *exactly* in this case, and a MCRG calculation always has an associated error. Evidence suggesting that this transition is first order has been presented by other authors [34]; still others [33] find a second-order transition with $\nu = 1/y = 0.3$.

A further interesting result is the fact that in our scheme the fixed point F is attractive along the line segment TF, implying that the critical behaviour along this segment is the same as that for the point F. This result is supported by an argument of Fradkin and Shenker [30], generalized by Shrock [24], which states that for small κ the model defined by eqs. (26) is equivalent to a pure U(1) gauge model with

$$\beta_{1,\mathrm{eff}} = \beta_1 + \tfrac{1}{4}\kappa^4 + \cdots\,. \tag{33}$$

There also exist in the flow diagram a tricritical point (labelled "T") and a fixed point (labelled "E"). In each case the exponents were determined by calculating the block couplings at each of 25 points forming a 5×5 square centered (approximately) at the fixed point. These couplings were measured over 500 iterations after 50 equilibration sweeps through the lattice. The results are as follows (note that, as per the discussion of eqs. (32), y_1 in eq. (34) is likely to be an *under*estimate):

$$\begin{aligned} T\colon\ \beta^* &= 1.1\,, \\ \kappa^* &= 0.4\,, \\ y_1 &\gtrsim 3.0\,, \\ y_2 &= 2.1\,; \end{aligned} \tag{34}$$

$$E:\ \beta^* = 0.4,$$
$$\kappa^* = 0.9,$$
$$y_{\mathrm{E}} = 3.\pm 1. \tag{35}$$

In the case of the fixed point E, our results suggest the existence of a phase transition along the axis $\beta_1 = 0$. Such a phase transition is not possible in the two-component (i.e. one *complex* component) abelian Higgs model [30, 31], for in that case the scalar field can be transformed away by a gauge rotation. A phase transition at vanishing β_1 has been observed numerically [24, 29] in this limit model. Arguments [24] based upon the obvious connection with an annealed spin model suggest that this transition is second order, implying [35] that y_{E} is less than four. At $\beta_1 = 0$, y is exactly 2 [24]: if our flow structure is correct, that implies that y at the fixed point E must be 2.

4.3. THE SU(2)-HIGGS MODEL

This model is the limit of the lattice standard model when β_1 is allowed to increase without bound. We have previously made a (very crude) MCRG study of this model [7]; here the results of an improved study are given. Nevertheless, it is our contention that this sector of the theory is the most difficult to understand quantitatively. Some of the problems associated with MCRG calculations are described below.

The model is described by an action

$$S_{\mathrm{SU2}} = S_2 + S_{\mathrm{L2}}, \tag{36a}$$

$$S_{\mathrm{L2}} = -2\kappa \operatorname{Re} \sum_{n_\mu} \Phi_n^+ V_{n,\mu} \phi_{n+\mu}, \tag{36b}$$

with the conventions established previously. The fundamental representation of SU(2) is employed for the link field V, which can then be written

$$V = a_0 + i\boldsymbol{a}\cdot\boldsymbol{\sigma}, \tag{37}$$

where the $\boldsymbol{\sigma}$ are the Pauli spin matrices,

$$\sigma_a \sigma_b = \delta_{ab} + i\varepsilon_{abc}\sigma_c \tag{38}$$

and a_μ is a four-component object satisfying the relation (so that V has unit determinant):

$$a^2 \equiv a_\mu a^\mu \equiv a_0^2 + \boldsymbol{a}^2 = 1. \tag{39}$$

The Haar measure for the group is then given by [37]

$$\mathrm{d}V = \frac{1}{2\pi^2}\delta(a^2 - 1)\,\mathrm{d}^4 a\,. \tag{40}$$

The three parameters α_p ($p = 1, 2, 3$) can be defined implicitly by

$$a_\mu = \left(\cos|\boldsymbol{\alpha}|,\ \frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}\sin|\boldsymbol{\alpha}|\right) \tag{41}$$

in accordance with the standard notation:

$$V = \exp(i\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})\,. \tag{42}$$

The sum of SU(2) elements V differs from another SU(2) element by at most an overall normalization factor. This serendipity suggests a definition of the blocking transformation for the gauge fields of the form

$$V'_{n,\mu} = \frac{V_{\text{sum}}}{\det V_{\text{sum}}}\,, \tag{43a}$$

where

$$V_{\text{sum}} \equiv \sum_{\text{paths}} V_{\text{path}} \tag{43b}$$

and V_{path} refers to the ordered product of V's along a given path between the two block sites connected by $V'_{n',\mu}$. The specific choice of paths is quite important for the SU(2)-Higgs model and is discussed in detail below.

The block scalar field is (as with the U(1)-Higgs model discussed above) defined as the parallel-transported average of all scalar fields in the block, normalized so that $\Phi'^{+}_{n'}\cdot\Phi'_{n'} = 1$. Where multiple paths of parallel transport exist, the arithmetic mean of all of the shortest paths is taken.

The equations for the effective block couplings are derived by a procedure analogous to that given in eqs. (14). Under an infinitesimal SU(2) rotation in group space,

$$\Phi'_n \to (1 + i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})\Phi'_n \equiv \Phi'_n + \Delta_n\Phi'_n\,, \tag{44a}$$

$$V'_{n,\mu} \to (1 + i\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma})V'_{n,\mu} \equiv V'_{n,\mu} + \Delta_{n,\mu}V'_{n,\mu}\,. \tag{44b}$$

Following the same procedure as for the U(1)-Higgs limit described above, the (effective) renormalized couplings κ' and β' are extracted by use of the relations

$$\int \mathscr{D}\Phi'\mathscr{D}\Phi'^{+}\mathscr{D}V'\Delta_n\left[(\Delta_n S'_{L2})\mathrm{e}^{-S'_{SU2}}\right]=0\,, \tag{45a}$$

$$\int \mathscr{D}\Phi'\mathscr{D}\Phi'^{+}\mathscr{D}V'\Delta_{n,\mu}\left[(\Delta_{n,\mu} S'_{2})\mathrm{e}^{-S'_{SU2}}\right]=0\,, \tag{45b}$$

which lead to a pair of simultaneous linear equations as before.

As a practical matter, it is useful to note that the scalar fields for both the site *and* block systems can be absorbed by redefining the $V_{n,\mu}$ and $V'_{n',\mu}$ through gauge transformations [7]. Thus the scalar fields need not be simulated directly, and the action for the site system can be written

$$S_{SU2}=S_2+\tilde{S}_{L2}\,, \tag{46a}$$

with

$$\tilde{S}_{L2}=-2\kappa\sum_{n,\mu}\mathrm{Tr}\,V_{n,\mu}\,. \tag{46b}$$

After the block links $V'_{n',\mu}$ are calculated (by first averaging the appropriate paths via eqs. (43) and then removing the explicit block scalars $\Phi'_{n'}$ by a gauge transformation [7]) the block action – within the context of the two-parameter truncation assumed here – also can be written in the form eqs. (46). The equations for the renormalized couplings implied by eq. (45a) are:

$$2\kappa'=\frac{\sum_n\langle b_{0,n}\rangle}{\sum_n\frac{1}{3}\langle \boldsymbol{b}_n\cdot\boldsymbol{b}_n\rangle}\,, \tag{47a}$$

where

$$b_{0,n}=\tfrac{1}{2}\mathrm{Tr}(W_n)\,,$$

$$\boldsymbol{b}_n=\tfrac{1}{2}i\,\mathrm{Tr}(\boldsymbol{\sigma}W_n)\,, \tag{47b}$$

and

$$W_n\equiv\sum_{\mu}\left[V'_{n-\mu,\mu}+(V'_{n,\mu})^{\dagger}\right]. \tag{47c}$$

(The average over n in eq. (47a) is inserted in order to increase statistics.) Equation (45b) yields

$$\beta'_2=\frac{\sum_{n,\mu}\langle 3p'_{0,n,\mu}-2\kappa'\boldsymbol{a}'_{n,\mu}\cdot\boldsymbol{p}'_{n,\mu}\rangle}{\sum_{n,\mu}\langle\boldsymbol{p}'_{n,\mu}\cdot\boldsymbol{p}'_{n,\mu}\rangle}\,, \tag{48a}$$

where

$$a'_{n,\mu} \equiv \tfrac{1}{2} i \operatorname{Tr}(\sigma V'_{n,\mu}), \tag{48b}$$

$$p'_{n,\mu} \equiv \tfrac{1}{2} i \operatorname{Tr}(\sigma Q_{n,\mu}), \tag{48c}$$

$$p'_{0,n,\mu} \equiv \tfrac{1}{2} \operatorname{Tr}(Q_{n,\mu}) \tag{48d}$$

$$Q_{n,\mu} \equiv \sum_{\pm\nu} V'_{n,\mu} V'_{n+\mu,\nu} (V'_{n+\nu,\mu})^{\dagger} (V'_{n,\nu})^{\dagger} \tag{48e}$$

and again the sum over n and μ is inserted in order to improve statistics.

In each of the following analyses, the configurations of the SU(2)-Higgs model were generated by an attempered heat-bath routine based upon ideas of Creutz [37]. The block couplings were measured on a $6^4 \to 2^4$ lattice over 500 iterations, following 50 equilibration sweeps. The results of a $9^4 \to 3^4$ blocking were virtually identical to those of the $6^4 \mapsto 2^4$ blocking, and the inclusion of gauge-invariant operators equivalent in the O(4) limit to second- and third-neighbour scalar-scalar couplings produced only small qualitative changes in the flow structure.

It remains to specify the paths included in the sum in eq. (43b). The simplest choice is to block by a scale factor of two and choose the paths in figs. 5a and 5b along with obvious permutations). This is a procedure originally advocated by Swendsen [38]. However, the renormalization group flow structure (fig. 6) generated by this recursion process is inconsistent with our two-parameter truncation. In particular, the separatrix (governed by the fixed point D with critical index $y_D \sim 1$) extends from the gaussian O(4) XY model fixed point G *through* the half-line $\beta_2 = 0$ and continues on to negative β_2. Yet obviously when β_2 vanishes the links are decoupled and no phase transition can occur.

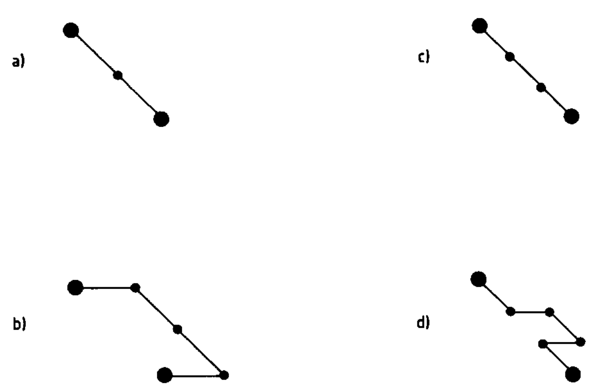

Fig. 5. Schema of paths included for various SU(2) blocking transformations: diagrams (a) and (b) show the Swendsen paths; diagrams (c) and (d) display our choice.

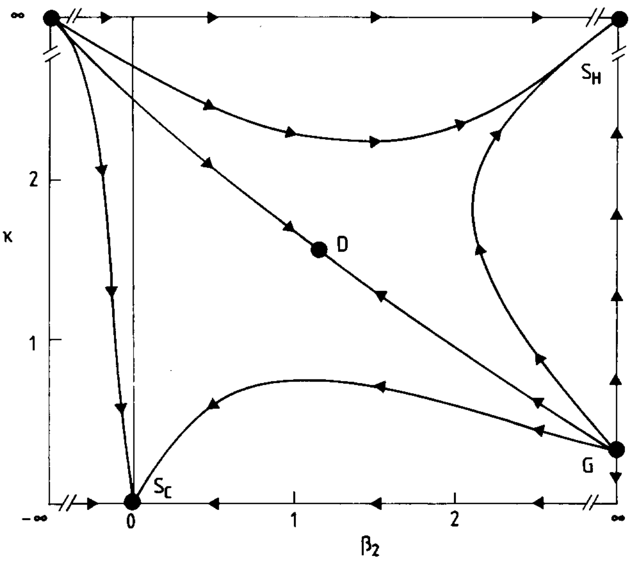

Fig. 6. Inconsistent flow diagram for the SU(2)-Higgs model generated with Swendsen paths.

The problem is that each site link appears in many distinct block links in the Swendsen procedure, and so this blocking method generates spurious interactions between block links when the site links are decoupled. One solution to this problem is to employ a blocking scheme proposed previously by us in ref. [17]. This scheme employs a blocking factor of 3 and uses only the paths displayed in figs. 5c and d. No spurious interactions are introduced, for no site link appears in more than one block link. A possible drawback of the method is that in four dimensions only one-fourth of the site links are included in the paths summed in eq. (43b), so a two-parameter truncation may prove inaccurate. Nevertheless, the flow structure (fig. 7) generated by this procedure agrees well with its known phase structure [39], which allows us some hope for a qualitative understanding of the model based upon our simple analysis.

The flow structure in our truncation scheme is depicted in fig. 7. Approximately 100 points were used to generate this flow diagram. The fixed point G is the usual gaussian phase transition of the O(4) XY model. We conjecture that another fixed point (labelled "M" in the figure) exists. This fixed point is presumably *marginal* ($y_1 = 0$) in the direction of the separatrix connecting it to point G. If the separatrix MG represents a phase transition, then the flows along it must terminate in a fixed point. The critical exponent which governs this phase transition can be determined by measuring the derivatives of the renormalized couplings in the direction *orthogonal* to this separatrix. Evidence which suggests that this separatrix represents a first-order phase transition has been presented [8]; if this is the case then [35] the

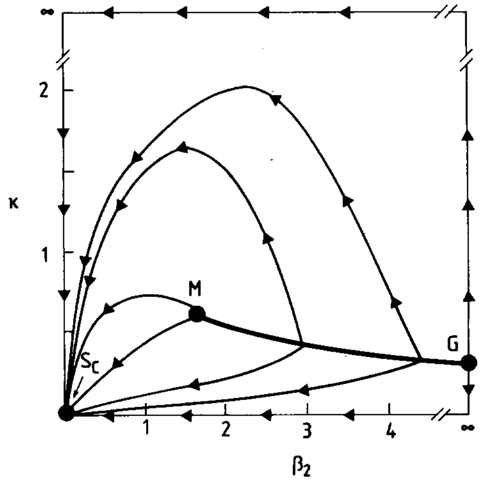

Fig. 7. Flow diagram for the SU(2)-Higgs model generated with our paths.

point M is a tricritical point with exponents $y_1 = 0$ and $y_2 = 4$. In our blocking scheme (with scale factor 3) this result would imply that the matrix $\partial K'_\alpha / \partial K_\beta$ has eigenvalues $3^{y_1} = 1$ and $3^{y_2} = 81$. It is difficult to measure a matrix with such disparate eigenvalues numerically, so our evidence for the existence of a marginal fixed point M is limited to an inference from the flow structure. The large scale factor $b = 3$ in use here clearly hampers the analysis. It is worth mentioning, however, that blocking transformations with a scale factor as small as $b = \sqrt{2}$ in two or four dimensions for lattice gauge theories can be constructed [16]. A more accurate study of the SU(2)-Higgs model should therefore be feasible.

Our conjecture that the point M is a marginal fixed point also implies that scaling violations could exist along the separatrix connecting M with the gaussian point G. The reasoning behind the idea [40] that marginal operators are connected with logarithmic scaling violations can be seen by considering the behaviour of two infinitesimal scaling fields μ_1 and μ_2 near a critical point $\mu_1^* = \mu_2^* = 0$. (For example, one might have $\mu_1 = (\beta_2 - \beta_{2_M}^*)/\beta_{2_M}^*$, $\mu_2 = (\kappa - \kappa_M^*)/\kappa_M^*$ if the MG separatrix was perfectly horizontal.) Suppose that under the rescaling of the lattice spacing $a \to a_l \equiv e^l a$ the fields $\mu_1(l)$ and $\mu_2(l)$ flow in a fashion described by the equations

$$\frac{\partial \mu_1(l)}{\partial l} = y_{11}\mu_1 - y_{111}(\mu_1)^2 + \cdots, \tag{49a}$$

$$\frac{\partial \mu_2(l)}{\partial l} = y_{22}\mu_2 - y_{212}\mu_1\mu_2 + \cdots. \tag{49b}$$

If y_{11} is greater than zero, the fixed point $\mu_1^* = \mu_2^* = 0$ has two relevant directions (i.e. is a tricritical point) with exponents y_1 and y_2:

$$\mu_1(l) = \mu_1(0)\mathrm{e}^{y_{11}l} \sim a_l^{y_{11}}, \tag{50a}$$

$$\mu_2(l) = \mu_2(0)\mathrm{e}^{y_{22}l} \sim a_l^{y_{22}}. \tag{50b}$$

If, however, y_{11} is *zero* (corresponding to a *marginal* field μ_1) then if $\mu_1(0)$ is positive it follows that for large l

$$\mu_1(l) \to \frac{1}{y_{111}l} \sim \frac{1}{\ln a_l}, \tag{51a}$$

$$\mu_2(l) \to \mu_2(0)\mathrm{e}^{y_{22}l} \cdot l^{\theta} \tag{51b}$$

$$\sim a_l^{y_{22}}(\ln a_l)^{\theta}, \tag{51c}$$

$$\theta \equiv y_{212}/y_{111} \tag{51d}$$

and eq. (52c) implies [40,41] that thermodynamic quantities which depend upon μ_2 possesses logarithmic scaling violations. No such scaling violations occur if $\mu_1(0) \leqslant 0$. It is interesting to note that the perturbative renormalization group predicts [41,40] logarithmic corrections to mean-field scaling in pure ϕ^4 theory; should these be present at the gaussian fixed point of the non-perturbative theory then (as shown by Aragão de Carvalho et al. [5]) triviality of the pure ϕ^4 theory is implied. No corresponding result is known for the SU(2)-Higgs model however, and we mention this point only to illustrate the potential importance of corrections to scaling.

5. Analysis of the SU(2) × U(1)$_Y$ standard model

The full lattice standard model (as defined by eqs. (10)) is now considered. The blocking schemes for the U(1) and SU(2) gauge fields are as described in subsects. 4.2 and 4.3 respectively. The block scalar field is defined as the normalized average of all scalar fields in the block, parallel transported with the U(1) and SU(2) gauge fields. As usual, where multiple paths of parallel transport exist, the arithmetic mean of all the shortest paths is taken.

The equations used to generate the block-renormalized couplings (as usual, assuming that the block action is described well by an effective action of precisely the same functional form as the site action) are determined by the invariance equations

$$\mathrm{Tr}\,\Delta_{n'}\left[(\Delta_{n'}S'_{\mathrm{L}})\mathrm{e}^{-S'}\right]=0\,, \tag{52a}$$

$$\mathrm{Tr}\,\Delta_{n',\mu}\left[(\Delta_{n',\mu}S'_{2})\mathrm{e}^{-S'}\right]=0\,, \tag{52b}$$

$$\mathrm{Tr}\frac{\delta}{\delta\theta'_{ij}}\left[\left(\frac{\delta}{\delta\theta_{ij}}S'_{1}\right)\mathrm{e}^{-S'}\right]=0\,, \tag{52c}$$

where the notation

$$\mathrm{Tr}\equiv\int\mathscr{D}U\int\mathrm{D}V\int\mathscr{D}\Phi^{+}\int\mathscr{D}\Phi \tag{52d}$$

has been used.

As a practical aside, we simulated the site system in a gauge where each of the complex scalar site fields had an upper component equal to unity and a zero lower component. The block scalar fields were calculated by the above-described parallel transport,

$$\Phi'=N^{-1}\sum_{\text{paths}}\left\{\left[\prod_{\text{path}}(UV)\right]\cdot\begin{pmatrix}1\\0\end{pmatrix}\right\} \tag{53a}$$

$$=\begin{pmatrix}\phi'_{+}\\ \phi'_{0}\end{pmatrix}, \tag{53b}$$

where N is chosen so that

$$|\phi'_{+}|^{2}+|\phi_{0}|^{2}=|\Phi'|^{2}=1\,. \tag{53c}$$

The block scalar fields can in general be eliminated by performing gauge rotations on the SU(2) and U(1) fields. With the gauge chosen here, it suffices to rotate only the SU(2) gauge fields:

$$V'_{n,\mu}\to W_{n}V'_{n,\mu}W^{+}_{n+\mu}\,, \tag{54}$$

where W_n is defined so that

$$W_{n}\Phi'_{n}=\begin{pmatrix}1\\0\end{pmatrix}, \tag{55a}$$

$$W^{+}_{n}W_{n}=\mathbb{1}\,, \tag{55b}$$

i.e.

$$W=\begin{pmatrix}(\phi'_{+})^{*} & (\phi'_{0})^{*}\\ -\phi'_{0} & \phi'_{+}\end{pmatrix}, \tag{55c}$$

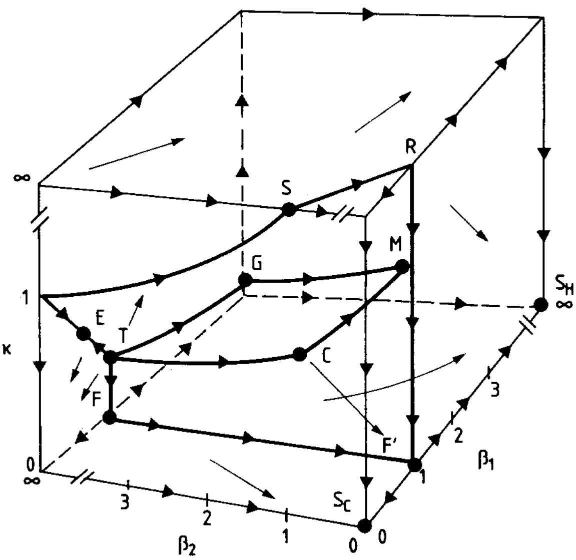

Fig. 8. Flow diagram for the lattice standard model.

so that

$$U'_{\square}, \tag{56a}$$

$$\mathrm{Tr}\, V'_{\square}, \tag{56b}$$

and

$$\Phi'^{+}_n U'_{n,\mu} V'_{n,\mu} \Phi'_{n+\mu} \tag{56c}$$

remain invariant.

Thus (as in the SU(2)-Higgs limit discussed in subsect. 4.3.) no explicit reference to the scalar fields appears in the calculation. Since expectation values of the various functionals used in the calculation of block couplings are *ratios*, the spurious integration over gauge degrees of freedom simply drops out.

The flow diagram for the lattice standard model is displayed in fig. 8. We used in this case approximately 500 points. It is consistent with the phase diagram found [24] by Shrock. Throughout the flow diagram β_2 decreases monotonically, and the fixed points T and E found in the SU(2)-Higgs model (fig. 7) merge into a new multicritical point C in the centre of the cube. This fixed point is evidently marginal ($y_3 = 0$) in essentially the β_2 direction, and is located at $(\kappa, \beta_1, \beta_2) \sim (0.9, 0.9, 1.6)$. The critical exponents y_1 and y_2 for C were determined by a double least-squares fit to 27 points, reported in table 1 (with 500 measurement iterations following 100 equilibration steps) in a plane located at $\beta_2 = 1.75$. The results are $y_1 = 1.3$, $y_2 = 0.03$. The measurement was repeated at $\beta_2 = 1.60$, the result is $y_1 = 1.4$,

TABLE 1
Projection in the $\beta_2 = 1.75$ plane of the coupling constant's flow around the point C

(β_1, k)	(β_1', k')
(1.0, 0.75)	(0.841, 0.82)
(0.85, 0.9)	(0.77, 1.12)
(0.85, 1.0)	(0.85, 1.45)
(0.8, 0.9)	(0.66, 0.94)
(0.8, 1.0)	(0.76, 1.33)
(0.8, 1.1)	(0.84, 1.71)
(0.75, 1.1)	(0.69, 1.56)
(0.9, 0.9)	(0.69, 0.73)
(0.9, 0.9)	(0.87, 1.21)
(0.85, 1.1)	(0.885, 1.69)
(0.95, 0.8)	(0.81, 0.87)
(0.95, 1.0)	(0.94, 1.44)
(0.95, 0.9)	(0.94, 1.29)
(0.9, 0.95)	(0.89, 1.38)
(0.85, 0.9)	(0.77, 1.12)
(0.9, 0.85)	(0.80, 0.94)
(0.875, 0.95)	(0.855, 1.32)
(0.875, 0.85)	(0.71, 0.875)
(0.875, 0.9)	(0.845, 1.2)
(0.85, 0.85)	(0.62, 0.72)
(0.85, 0.95)	(0.825, 1.29)
(0.875, 0.9)	(0.845, 1.20)
(0.925, 0.85)	(0.83, 1.03)
(0.925, 0.9)	(0.92, 1.31)
(0.925, 0.95)	(0.93, 1.37)
(0.875, 0.85)	(0.71, 0.875)
(0.875, 0.95)	(0.855, 1.32)

$y_2 = -0.02$. As suggested by a slice (fig. 9) of the flow diagram, the fixed point C is *doubly* marginal in essentially the κ and β_2 directions.

The flow diagram for the three-parameter theory possesses two main sinks – the sink S_C for the confinement phase (located at $(\kappa, \beta_1, \beta_2) = (0,0,0)$), and the sink S_H for the Higgs phase (located at $(\kappa, \beta_1, \beta_2) = (0, \infty, 0)$). It has been shown [24] that the line segment SR corresponds to the phase transition at $\beta_1 \sim 1$ in a pure fundamental representation U(1) gauge theory. Our flow diagram predicts that this phase transition is governed by the fixed point F′, which is a pure U(1) transition (fig. 4).

Our most interesting result is the fact that for finite β_2 *no* fixed points with *three* relevant directions appear. This finding suggests that (with the possible exception of a continuum limit obtained from an approach to the point C from the direction of small β_2) a non-trivial continuum limit of the theory possesses at most two independent renormalized parameters.

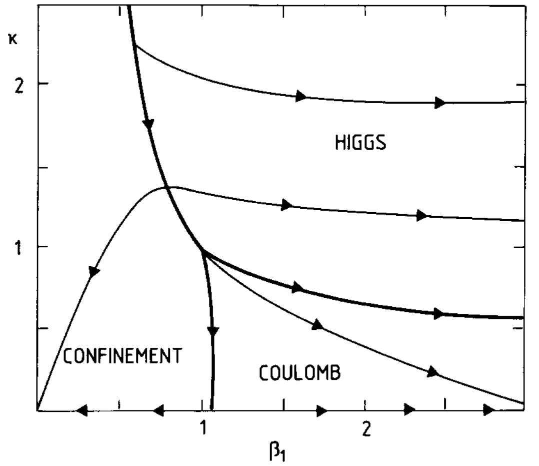

Fig. 9. "Snapshot" of fig. 8 when $\beta_2 = 1.75$.

The reasoning [41] behind this connection between the number of independent renormalized parameters and the relevant directions of a fixed point can be summarized crudely as follows. Under a blocking transformation, the lattice spacing increases. Thus, in order to take a continuum limit (by *decreasing* the lattice spacing) one must follow the renormalization group flows given here *backwards* while keeping the renormalized parameters of the continuum theory fixed. In order thus to approach a fixed point while holding n renormalized couplings fixed, it must be so that the fixed point has at least n relevant directions.

It may be that a continuum limit exists at the fixed point C when it is approached from the direction $\beta_2 < \beta_{2C}^*$. Such an approach implies that β_2 is *bounded*, so that corresponding bounds on the renormalized parameters of the $\beta_{2,R}$ possesses an upper bound, an upper bound on the Higgs mass is implied [11,12]. If *no* continuum limit is reached by approaching C in this direction, our analysis suggests that any non-trivial continuum theory of this model has a *calculable* Higgs mass.

Several important assumptions of this work must be kept in mind. First, ours is a *truncated* calculation – we have not demonstrated that additional relevant operators do not exist in the fixed length limit. However, it would be somewhat surprising if this was the case, given the accuracy of the flow diagram and critical exponents in comparison with the phase diagram and other known results (summarized in ref. [24]).

A second and more important point is the fact that we work with a model where the scalar field has a fixed length. This is equivalent to studying the theory at a fixed (infinite) value of the bare quartic coupling constant. If at the appropriate fixed points in the full parameter space (where the quartic coupling is allowed to vary) the quartic coupling is relevant, no further constraints are likely to be generated.

However, if the quartic coupling is always irrelevant, then even if the continuum limit of the theory is reached by approaching fixed point C from the direction of $\beta_2 < \beta_2^*(\mathrm{C})$, the number of renormalized parameters of the theory is less than the number of bare parameters and the Higgs mass should be predictable at such a continuum limit.

Another important assumption of our work is the fact that all our conclusions are based upon the results of a single blocking transformation. Unfortunately (and this is a serious drawback of the present study) the blocking transformation we applied has such a large scale factor ($b = 3$) that a second blocking (from a $(18)^4$ lattice) was unpractical. We are presently developing methods involving smaller scale factors [43].

Finally, it should be mentioned that this calculation makes no statements about the validity of hyperscaling in the full $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ model, for it is impossible to check hyperscaling within the framework of the present Monte Carlo renormalization group [42].

6. Summary and conclusions

We have presented the results of the first Monte Carlo renormalization group study of the lattice $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ standard model (minus fermions). Within a well-defined minimal truncation scheme, the flow diagram, fixed points, and critical exponents were calculated. A most remarkable result was found – in our scheme there are no fixed points at finite β_2 with three relevant directions, suggesting that if a non-trivial continuum theory exists then the Higgs mass may be bounded or predictable.

Like any numerical simulation, ours relies upon specific and systematically improvable assumptions. Foremost amongst these is the truncation of the flow equations to a space of three couplings. We have outlined in detail the procedure by which this approximation can be improved to arbitrary accuracy; further calculations should certainly be performed. Nevertheless, we point out that our flow diagram and resultant critical exponents imply (and therefore agree with) many known results. Most of the remaining uncertainties are well-known to numerical analysts. For example, it is in general impossible to prove that a simulation has converged or that the infinite-volume limit has been reached. Although numerical calculations such as ours cannot therefore even in principle provide a proof of a result, they can produce evidence for its validity and, we hope, inspire further interest in the problem.

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