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TOPICS IN LATTICE GAUGE THEORY*

Lectures for Non-Particle Theorists Given to the Materials Science and Technology Division at Argonne National Laboratory

David J. E. Callaway
High Energy Physics Division
Argonne National Laboratory
Argonne, Illinois 60439

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OUTLINE

Prolegomena

Throughout these lectures I shall use the following notation:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} ,$$

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z ,$$

$$d^4x \equiv dx\ dy\ dz\ dt \ ;$$

while units are chosen so that $\hbar = c = 1$. Summation is implicit,

$$A^\mu B_\mu = A^\mu B^\nu g_{\mu\nu}$$

$$= \sum_{\mu,\nu=0}^{3} A^\mu B^\nu g_{\mu\nu} ,$$

where $g^{\mu\nu}$ is the metric tensor,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

The momentum operator in coordinate representation is given by

$$p_\mu = -i\,\partial_\mu ,$$

$$p^\mu p_\mu = -\partial^\mu \partial_\mu \equiv -\Box ,$$

The four-vector potential for the electromagnetic field is defined by

$$A^{\mu} = (\Phi, \vec{A}) \ .$$

The field strengths are then

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \ ,$$

while the electric and magnetic fields are given by

$$E = (-F^{01}, -F^{02}, -F^{03}) \ ,$$

$$B = (-F^{23}, -F^{31}, -F^{12}) \ .$$

Functional differentiation is defined by

$$\frac{\delta}{\delta\phi(y)} \int_{-\infty}^{\infty} dx \ F(x) \ \phi(x) = F(y) \ .$$

Other conventions follow Bjorken and Drell[1].

I. CONTINUUM GAUGE THEORIES

A. From a Symmetry Principle to a Lagrangian

Before discussing lattice gauge theories, it is instructive to review briefly the concepts of continuum gauge theories. More detailed discussions can be found in Abers and Lee[2], Taylor[3], Kibble[4], and elsewhere.

A fundamental ingredient of any quantum field theory (in the continuum or on a lattice) is the action, which is given by the time integral of the Lagrangian:

$$S = \int_{-\infty}^{\infty} L \; dt = \int d^4x \; \mathscr{L}\{\phi, \partial_\mu \phi\} \; . \tag{I.1}$$

The Lagrangian is in turn given by the space integral of the Lagrangian density $\mathscr{L}$, which is a functional of the fields $\{\phi\}$ and their derivatives $\{\partial_\mu \phi\}$. The classical equations of motion of this theory are derived from Hamilton's principle:

$$\delta \int \mathscr{L} dt = 0 \; . \tag{I.2}$$

Equation (I.2) implies that the Lagrangian density must obey Euler's equations,

$$\frac{\delta \mathscr{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathscr{L}}{\delta(\partial_\mu \phi)} \; . \tag{I.3}$$

A simple example is the Lagrangian density of a noninteracting scalar field,

$$\mathscr{L}\{\phi,\partial_\mu\phi\} = \frac{1}{2}\left[(\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi\right] , \tag{I.4}$$

where m is the mass of the charged quanta associated with the quantum theory. The Euler equation of motion for this Lagrangian density is

$$(\partial_\mu\partial^\mu + m^2)\phi \equiv (\Box + m^2)\phi = 0 , \tag{I.5}$$

which is the familiar* Klein-Gordon equation for a spinless particle of mass m.

For every continuous symmetry of the Lagrangian there corresponds a conservation law. Conversely, for every conserved quantum number there exists a transformation on the fields of the theory which leaves the Lagrangian density invariant. The simplest example of this general phenomenon involves electric charge. Consider a group of transformations on the fields ϕ

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*For those unfamiliar with the Klein-Gordon equation, it may seem less mysterious when written in component form:

$$(\partial^2/\partial t^2 - \vec{\nabla}^2 + m^2)\phi = 0 ,$$

which, for $m = 0$, is just the standard wave equation. Then if the usual quantum mechanical identification

$$-i\partial/\partial t \to E$$

$$-i\vec{\nabla} \to \vec{p}$$

is made we see that the Klein-Gordon equation arises from the relativistic kinematic requirement

$$E^2 - \vec{p}^2 = m^2 .$$

$$\phi(x) \rightarrow e^{-iq\theta}\phi(x) ,$$
$$\phi^*(x) \rightarrow e^{+iq\theta}\phi^*(x) \tag{I.6}$$

where q is the charge of the field ϕ and θ parameterizes the transformation. The Lagrange density Eq.(I.5) is invariant under this set of transformations, which form the group of unitary transformations in one dimension [denoted by U(1)]. Note that even though contains gradients $\partial_\mu\phi$ of the fields, these terms are invariant under the transformation Eqs.(I.6) since θ is independent of x [i.e., Eqs.(I.6) describe a global gauge transformation].

For infinitesimal θ (=$\delta\theta$), the global gauge transformation Eqs.(I.6) reads

$$\delta\phi = -i(\delta\theta)q\phi ,$$
$$\delta\phi^* = +i(\delta\theta)q\phi^* . \tag{I.7}$$

The condition that the Lagrangian density $\mathscr{L}$ be invariant under this transformation can be written

$$\delta\mathscr{L} = 0 = \frac{\delta\mathscr{L}}{\delta\phi}\delta\phi + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$
$$+ \frac{\delta\mathscr{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi^*)}\delta(\partial_\mu\phi^*) , \tag{I.8a}$$

$$= - i\delta\theta\, \partial_\mu\left[\frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi)} q\phi + \frac{\delta\mathscr{L}}{\delta(\partial_\mu\phi^*)} q\phi^*\right] . \tag{I.8b}$$

From Eq.(I.8) it can be seen that the current

$$J^{\mu} \equiv i\left[\frac{\delta \mathscr{L}}{\delta(\partial_{\mu}\phi)}\, q\phi - \frac{\delta \mathscr{L}}{\delta(\partial_{\mu}\phi^{*})}\, q\phi^{*}\right] , \tag{I.9a}$$

is conserved, that is:

$$\partial_{\mu} J^{\mu} = 0 . \tag{I.9b}$$

In our simple example Eq.(I.5):

$$J^{\mu} = i(\phi\partial_{\mu}\phi^{*} - \phi^{*}\partial_{\mu}\phi) , \tag{I.10}$$

and in the corresponding second quantized theory the operator Q

$$Q = \int d^{3}x\, J_{0} , \tag{I.11}$$

is the charge operator.

We have seen that the conservation law (Eq.I.9) arises because the Lagrangian density is invariant under a global change of the phase of the fields $\phi(x)$. In other words, it is possible to redefine the phase of the field $\phi(x)$ by an additive constant, provided that this constant does not depend on x.

It seems somewhat peculiar however that the phase convention chosen at one point should constrain the choice of convention at all the points of spacetime. Such a concept does not appear to be consistent with the localized field concept that underlies the usual physical theories (Yang and Mills[5]).

If a transformation of the form Eq.(I.6) but with $\theta(x)$ taken to be a real function of x is applied to the field $\phi(x)$ it will transform as

$$\phi(x) \rightarrow \exp[-iq\theta(x)]\phi(x) \ . \tag{I.12}$$

Terms in the Lagrangian which depend only on the fields themselves are invariant under this local gauge transformation, e.g.,

$$m^2\phi^*\phi \rightarrow m^2\phi^*\phi \ . \tag{I.13}$$

However, terms involving the gradients of the fields $\partial_\mu\phi$, such as the kinetic energy term, are not invariant:

$$\partial_\mu\phi \rightarrow \exp[iq\theta(x)]\partial_\mu\phi - iq[\partial_\mu\theta(x)]\exp[iq\theta(x)]\phi(x) \ . \tag{I.14}$$

The Lagrangian density can however be made invariant under a local gauge transformation by the introduction of the electromagnetic field in a fashion usually referred to as minimal coupling. This procedure involves the replacement of the gradient operator ∂_μ with

$$\partial_\mu - ieqA_\mu(x) \equiv D_\mu \ , \tag{I.15}$$

in the Lagrangian. If, under the local gauge transformation Eq.(I.12), $A_\mu(x)$ transforms in the fashion

$$A_\mu(x) \rightarrow -\frac{1}{e}\partial_\mu\theta(x) + A_\mu(x) \ , \tag{I.16}$$

then the covariant derivative $D_\mu\phi(x)$ transforms as follows:

$$D_\mu\phi(x) \equiv [\partial_\mu - ieqA_\mu(x)]\phi(x) \rightarrow \exp[-iq\theta(x)]D_\mu\phi(x) \quad . \qquad (I.17)$$

There should, of course, be quadratic kinetic energy terms in the Lagrangian which couple A_μ only to itself. It is easy to show that

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (I.18)$$

is invariant under the transformation Eq.(I.16). Thus the scalar combination

$$\mathscr{L}_{EM} = -\frac{1}{4} F_{\mu\nu}(x)F^{\mu\nu}(x) \qquad (I.19)$$

is an acceptable Lagrangian density for pure electromagnetism (the factor -1/4 is included by convention).

The transformation laws Eq.(I.16) are, of course, very familiar as the "gauge transformation" of electrodynamics. What is new is the derivation of the Lagrangian density of a gauge theory from a simple concept--the extension of a global symmetry of the Lagrangian to a local one. As we will see, this concept can be applied successfully to many types of symmetry. In particle physics, it is important to consider groups other than U(1). If we consider a general internal symmetry Lie group G, a gauge transformation can be written as

$$\phi \rightarrow \exp(iL\cdot\theta)\phi \quad . \qquad (I.20)$$

In Eq.(I.12), the ϕ can be taken as a column vector and the L are an appropriate matrix representation of the group G. These representation matrices satisfy the commutation relations

$$L_i L_j - L_j L_i \equiv [L_i, L_j] = i c_{ijk} L_k , \tag{I.21}$$

where the c_{ijk} are the structure constants of the group. Groups with $c_{ijk} \neq 0$ are called non-Abelian. For example, in the case of the group SU(2),

$$c_{ijk} = \varepsilon_{ijk} , \tag{I.22}$$

where $\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} = -\varepsilon_{kji} = 1$ is the totally anti-symmetric Levi-Civita tensor. One particular representation (the "fundamental" representation) of SU(2) can be given in terms of the three Pauli matrices τ_i,

$$L_i = \frac{1}{2} \tau_i . \tag{I.23}$$

It is possible to construct a covariant derivative $D_\mu \phi(x)$,

$$D_\mu \phi(x) \equiv [\partial_\mu - igL \cdot A_\mu(x)] \phi(x) \tag{I.24}$$

such that under the local gauge transformation Eq.(I.20) with θ a function of x,

$$D_\mu \phi(x) \rightarrow \exp[iL \cdot \theta(x)] D_\mu \phi(x) , \tag{I.25}$$

which requires

$$A^i_\mu \to A^i_\mu - \frac{1}{g}\,\partial_\mu(\delta\theta)^i + c_{ijk}(\delta\theta)^j A^k_\mu \ , \qquad (I.26)$$

for infinitesimal $\delta\theta$.

For this general non-Abelian gauge theory the field strength tensor is

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g c_{ijk} A^j_\mu A^k_\nu \ , \qquad (I.27)$$

where i, j, and k refer to group indices. Under the local gauge transformation Eq.(I.20),

$$F^i_{\mu\nu} \to F^i_{\mu\nu} + c_{ijk}(\delta\theta^j)F^k_{\mu\nu} + o(\delta\theta)^2 \ . \qquad (I.28)$$

Thus, if the pure gauge field Lagrangian density is chosen to be

$$\mathscr{L}_0 = -\frac{1}{4}\,\mathrm{Tr}\left(F^i_{\mu\nu} F^{\mu\nu\, i}\right) \qquad (I.29)$$

then it is invariant under a local gauge transformation.

We see that given an appropriate group we can construct a gauge-invariant classical Lagrangian. In the next section we will see how to construct a quantum theory, given a classical Lagrangian.

B. From a Lagrangian to a Quantum Field Theory

The method of path-integral quantization is developed below following Abers and Lee[2]. First, a path integral representation of the Schwinger transformation function is given. Denote by $Q(t)$ the position operator in the Heisenberg picture, with eigenstates $|q,t\rangle$:

$$Q(t)|q,t\rangle = q|q,t\rangle \ . \tag{I.30}$$

The transformation function, which is the probability amplitude for finding a particle at point q' at time t' given that it was at point q at time t is given by

$$F(q',t';\ q,t) = \langle q',t'|q,t\rangle \ . \tag{I.31}$$

The transformation function can also be written in terms of the time-independent eigenstates of the Schrodinger picture. The position operator in this picture is denoted by Q_s, and is related to the Heisenberg picture operator by

$$Q(t) = \exp(iHt)Q_s\exp(-iHt) \ , \tag{I.32}$$

where the Hamiltonian H is time-independent. The eigenstates associated with the operator Q_s are denoted by $|q\rangle$:

$$Q_s|q\rangle = q|q\rangle \ , \tag{I.33}$$

with

$$|q\rangle = \exp(-iHt)|q,t\rangle \ , \tag{I.34}$$

so that

$$F(q',t';\ q,t) = \langle q'|\exp[iH(t-t')]|q\rangle \ . \tag{I.35}$$

Now split up the time interval from t to t' into a large number n + 1 of intervals of duration ε. Then, by completeness,

$$F(q',t';\ q,t) = \langle q,t+(n+1)\varepsilon\,|q,t\rangle$$

$$= \int dq_1 \dots dq_n \langle q',t'|q_n,t_n\rangle\langle q_n,t_n|q_{n-1},t_{n-1}\rangle \tag{I.36}$$

$$\times \langle q_{n-1},t_{n-1}|q_{n-2},t_{n-2}\rangle \dots \langle q_1,t_1|q,t\rangle \ .$$

In the limit $\varepsilon \to 0$, $n \to \infty$,

$$\langle q_m,t_m|q_{m-1},t_{m-1}\rangle = \langle q_m|\exp(-i\varepsilon H)|q_{m-1}\rangle$$

$$= \delta(q_m - q_{m-1}) - i\varepsilon\langle q_m|H|q_{m-1}\rangle \ . \tag{I.37}$$

If the Hamiltonian H(P,Q) can be written as a function of P plus a function of Q [for example, if $H = P^2/2 + V(Q)$] then

$$\langle q_m | H | q_{m-1} \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp[ip(q_m - q_{m-1})] H\left(p, \frac{q_m + q_{m-1}}{2}\right) . \qquad (I.38)$$

Therefore, taking the Fourier transform of the delta function,

$$\langle q_m, t_m | q_{m-1}, t_{m-1} \rangle = \int \frac{dp}{2\pi} \exp\{i[p(q_m - q_{m-1}) - \varepsilon H(p, \frac{q_m + q_{m-1}}{2})]\} + O(\varepsilon^2) . \qquad (I.39)$$

The transformation function then becomes

$$F(q',t';\ q,t) = \lim_{n\to\infty} \prod_{i=1}^{n} \int dq_i \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} \times \exp\{i \sum_{j=1}^{n+1} [p_j(q_j - q_{j-1}) - \frac{t'-t}{n+1} H(p_j, \frac{q_j + q_{j-1}}{2})]\} \qquad (I.40)$$

which in the limit $n \to \infty$ is the operational definition of the path integral

$$F(q',t';\ q,t) = \int \frac{\mathscr{D}p\,\mathscr{D}q}{2\pi} \exp\{i\int_t^{t'} [p\dot{q} - H(p,q)]dt\} . \qquad (I.41)$$

If $H(P,Q) = \frac{1}{2} P^2 + V(Q)$, then, by use of the Fresnel integral

$$\frac{1}{\sqrt{2\pi i\varepsilon}} \exp(\tfrac{1}{2} i\varepsilon q^2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp[i\varepsilon(p\dot{q} - \frac{p^2}{2})] , \qquad (I.42)$$

it follows that the transformation function can be written in terms of the classical Lagrangian:

$$F(q',t';\ q,t) = \int \frac{\mathscr{D}q}{\sqrt{2\pi i\varepsilon}} \exp[i\int_t^{t'} \mathscr{L}\{q,\dot{q}\}\ dt] . \qquad (I.43)$$

Similarly, the expectation values of various operators can be expressed in terms of path integrals. For example, if a time-ordering operator T is defined by

$$T[Q(t_1)Q(t_2)] = \begin{cases} Q(t_1)Q(t_2) & (t_1>t_2) \\ Q(t_2)Q(t_1) & (t_2>t_1) \end{cases} , \qquad (I.44)$$

then it follows that the expectation value of the time-ordered Greens function is given by:

$$\langle q',t'|T[Q(t_1)Q(t_2)]|q,t\rangle = \int \frac{\mathcal{D}q}{\sqrt{2\pi i\varepsilon}}\, q(t_1)q(t_2)\exp[\, i \int_t^{t'} \mathcal{L}(q,\dot{q})dt] \;. \qquad (I.45)$$

The expectation value of operators in the ground state (i.e., the vacuum expectation values of the theory) are calculated by taking the limit $t \to -\infty$, $t' \to \infty$. By making a Wick rotation ($t \to -i\tau$) to imaginary times, it follows that

$$\langle 0|T[Q(t_1)\cdots Q(t_n)]|0\rangle = \int \frac{\mathcal{D}q}{\sqrt{2\pi\varepsilon}}\, q(t_1)\cdots q(t_n)\exp[-\int_{-\infty}^{\infty} d\tau\, \mathcal{L}_E(q,\dot{q})] \;, \qquad (I.46)$$

where $\mathcal{L}_E$ is the Lagrangian density in Euclidean space. Note the resemblance between Eq.(I.46) and the canonical ensemble of classical statistical mechanics. This correspondence will prove to be very useful in the following discussion.

II. GAUGE THEORIES ON A LATTICE

A. Motivation

In the last section the structure of gauge theories in the continuum was presented. In this section, true to the title of these lectures, they will be formulated on a lattice. It is first appropriate to review some of the reasons for approaching quantum field theory in this fashion.

Historically the motivation (Wilson[6]) for formulating field theory on a lattice was in order to study a local non-Abelian gauge theory known as quantum chromodynamics, or QCD. This theory, which can be derived by the application of the techniques of the last section to the group SU(3), is at present the best candidate for a theory of the strong interaction. It is widely believed that hadrons (neutrons, protons, pions, etc.) consist of subunits called quarks, which are bound together by the forces of QCD (called gluons).

One important property which hadrons seem to possess is called <u>confinement</u>. If quarks are confined inside hadrons, then it is impossible to break hadrons apart and separate the quarks. Despite a paucity of precise predictions, the conviction is growing amongst particle physicists that confinement does indeed occur. It would be useful to be able to derive confinement from QCD. However QCD is a difficult theory to solve and can at present be analyzed only perturbatively in extreme kinematic regions.

Unfortunately perturbation theory, generally the most useful tool of the field theorist, has failed to give any clear signals of confinement in QCD. In fact renormalization group arguments can be used to suggest that at large distances the effective quark-gluon coupling constant in QCD becomes large, so any perturbative expansion must break down. Clearly what is needed is a systematic, non-perturbative method of analyzing field theory. Such a method must surmount several obstacles, however.

One important problem of quantum field theory is the presence of ultraviolet infinities in the theory. Traditionally these infinities are handled by a procedure known as renormalization. If a theory is renormalizable, all infinite quanitities can be absorbed into a few parameters of the theory--such as the electric charge--which are replaced by their experimentally determined values. In perturbation theory, a multitude of renormalization techniques exist. These techniques usually rely upon the introduction of a high momentum (small distance) cutoff to render all quantities finite. The final stage in a perturbative calculation is to take the limit as the cutoff is removed; physical quantities such as the magnetic moment of the electron then approach a value which is independent of the renormalization prescription used.

The lattice formulation of quantum field theory provides an excellent framework for handling ultraviolet infinities. On a lattice, wavelengths less than the order of magnitude of the lattice spacing a are meaningless, and so therefore are momenta greater than $\sim o(\hbar/a)$. This cutoff is defined without reference to perturbation theory, and so is ideal for non-perturbative calculations. Lattice field theories are also amenable to numerical analysis, and thus may assist our understanding of nonperturbative aspects of field theory (such as confinement).

B. Compact Electrodynamics [U(1) Lattice Gauge Theory]

Probably the simplest lattice gauge theory with which to begin our discussion is the theory of quantum electrodynamics (QED) without matter fields on a lattice. In the continuum, this theory is completely soluble and describes free photons. Such a lattice theory presumably approaches the continuum theory in the limit in which the photon wavelength is large compared to the lattice spacing.

Recall that in the continuum Euclidean theory expectation values of field operators $\mathcal{O}$ are defined by

$$\frac{\langle 0|\mathcal{O}|0\rangle}{\langle 0|0\rangle} \qquad \langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A \; \mathcal{O}\{A\} \exp[-\int d^4x \; F_{\mu\nu}F^{\mu\nu}]}{\int \mathcal{D}A \; \exp[-\int d^4x \; F_{\mu\nu}F^{\mu\nu}]} , \qquad \text{(II.1)}$$

where

$$\mathcal{D}A = \prod_x \prod_\mu dA_\mu(x) \; . \qquad \text{(II.2)}$$

In order to discuss how the Wilson formulation of lattice gauge theories is related to the continuum theory it is useful to consider a four dimensional version of Stokes' theorem. Let S be a surface element and $dS_{\mu\nu}$ be an infinitesimal element embedded in this surface. Stokes' theorem states that the line integral of A_μ around S is equal to the surface integral of $F_{\mu\nu}$:

$$\oint_S A^\mu d\ell_\mu = \int_S dS_{\mu\nu} F^{\mu\nu} , \qquad \text{(II.3)}$$

where

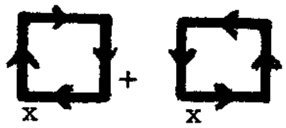
$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \quad . \tag{II.4}$$

Consider the quantity

$$\mathscr{L}(x) = \frac{1}{2a^4} \sum_{\square} \left[1 - e^{i \oint_P A^{\mu}(x) d\ell_{\mu}} \right] \tag{II.5}$$

where the sum over "plaquettes" $\square$ refers to a sum over the d(d-1) orthogonal square surfaces of size a^2 whose lower left hand corner is at the point x, for example, in two dimensions,

$$\sum_{\square} =$$



(II.6)

In four dimensions, there are a total of twelve such plaquettes per spacetime point. Stokes' theorem implies that,

$$\mathscr{L}(x) = \frac{1}{2a^4} \int_{S(x')} \frac{d^4x'}{a^4} \left[6 - \cos a^2 E_x - \cos a^2 E_y - \cos a^2 E_z - \cos a^2 B_x - \cos a^2 B_y - \cos a^2 B_z \right] \quad . \tag{II.7}$$

In the limit where the lattice spacing $a \to 0$,

$$\mathscr{L}(x) \to \frac{1}{2} \left[\vec{E}^2(x) + \vec{B}^2(x) \right] \quad , \tag{II.8}$$

which is the Euclidean action for QED in the continuum. Following Wilson the gauge fields are rescaled by the bare lattice coupling constant g_0,

$$A^{\mu}(x) \rightarrow \frac{1}{g_0} A^{\mu}(x) \qquad \text{(II.9)}$$

and so the QED continuum Lagrangian density becomes

$$\mathscr{L}(x) = \frac{1}{2g_0^2} \left[\vec{E}^2(x) + \vec{B}^2(x) \right] . \qquad \text{(II.10)}$$

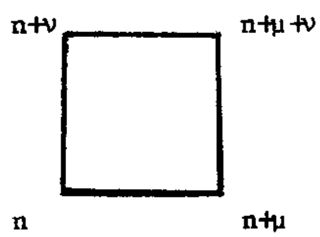
The above discussion is meant to provide some motivation for proposing a lattice Lagrangian density[6,7] which is a discrete version of its continuum counterpart Eq.(I.19):

$$S_{\text{lattice}} = \frac{1}{2g_0^2} \sum_{n,\mu\neq\nu} \left(1 - U_{n,\mu} U_{n+\mu,\nu} U^{-1}_{n+\nu,\mu} U^{-1}_{n,\nu}\right) , \qquad \text{(II.11)}$$

where

$$U_{n,\mu} \equiv \exp[iaA^{\mu}(n)] , \qquad \text{(II.12)}$$

and $U^{-1} = U^{+}$ is the Hermitian conjugate of U. The notation $A^{\mu}(n)$ refers to the variable located at the link leaving site n in direction μ. Thus the sum in Eq.(II.11) is over all plaquettes whose lower left corner is at the site n:



Each of the $A^{\mu}(n)$ is an independent variable. Because of the periodicity of the complex exponential, each field is integrated over a finite range (i.e. the theory is compact):

$$-\pi < aA^{\mu}(n) < \pi \ , \tag{II.13}$$

so that

$$-\frac{\pi}{a} < A^{\mu}(n) < \frac{\pi}{a} \ . \tag{II.14}$$

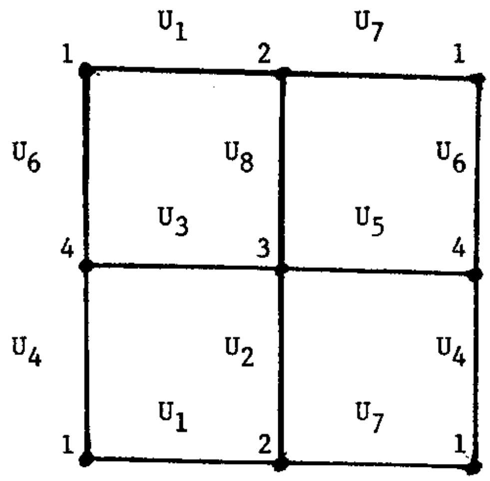
Note that in the limit $a \to 0$ the range of integration increases without bound and the theory becomes noncompact in the continuum limit, as it should. Also note that the action Eq.(II.11) is invariant under the local gauge transformation

$$U_{n,\mu} \to V_n U_{n,\mu} V^{+}_{n+\mu} \tag{II.15}$$

where the V_n are elements of the group U(1), i.e.,

$$V_n V_n^{+} = 1 \ . \tag{II.16}$$

It is instructive at this point to consider the theory of compact electrodynamics on a periodically continued 2×2 lattice. This theory is soluble in closed form, and will assist in the specification of conventions and other details. There are eight independent links connecting four spacetime points. The points are labelled 1, 2, 3, and 4; the connecting links are as shown:



If each of the U_n is parameterized in the fashion

$$U_n \equiv e^{i\phi_n} , \qquad \text{(II.17)}$$

where the ϕ_n are real fields ranging from $-\pi$ to π, then the action S is given by:

$$S = 4 - \mathrm{Re}\Big[e^{i(\phi_1 + \phi_2 - \phi_3 - \phi_4)} + e^{i(\phi_7 + \phi_4 - \phi_5 - \phi_2)} + e^{i(\phi_3 + \phi_8 - \phi_1 - \phi_6)} + e^{i(\phi_5 + \phi_6 - \phi_7 - \phi_8)}\Big]. \qquad \text{(II.18)}$$

The partition function of this system is defined by

$$Z \equiv \left(\prod_{i=1}^{8} \int_{-\pi}^{\pi} \frac{d\phi_i}{2\pi} \right) e^{-\beta S} ,$$

$$\beta \equiv 1/g_0^2 , \qquad \text{(II.19)}$$

where g_0 is the bare lattice coupling constant and the average plaquette is given by

$$P = \frac{1}{4} \langle S \rangle = - \frac{\partial}{\partial \beta} \ell n \; Z \; . \qquad \text{(II.20)}$$

Because of the local gauge symmetry in the action (and the periodic boundary conditions) there are only three linearly independent combinations of variables in the action, and thus really only three integrations to perform. If the integrations over the variables ϕ_1 and ϕ_7 are performed, the result is

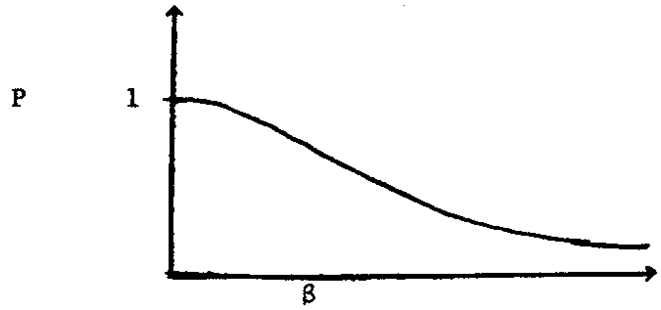
$$e^{-4\beta} I_0^2 [2\beta \; \cos \frac{1}{2} (\phi_2 - \phi_4 + \phi_6 - \phi_8)] \equiv e^{-S'} , \qquad \text{(II.21)}$$

so it follows that

$$Z = \int_{-\pi}^{\pi} \frac{d\phi_2}{2\pi} e^{-S'}$$

$$= e^{-4\beta} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} I_0^2 (2\beta \; \cos \theta) \; . \qquad \text{(II.22)}$$

It is easy to see that P ranges smoothly from unity (at $\beta = 0$) to zero (as $\beta \to \infty$). Since the system under consideration is finite, there are no phase transitions; and thus P is analytic for all β:



Let us now consider compact electrodynamics on a hypercube of side N lattice sites embedded in d dimensions. There are $\frac{1}{2}$ d(d-1) plaquettes per lattice site; thus, the average plaquette is given by,

$$P = N^{-d} \frac{2}{d(d-1)} \left(- \frac{\partial}{\partial\beta} \ell n\ Z \right) \ , \qquad \text{(II.23)}$$

where,

$$Z \equiv \left(\prod_i \int_{-\pi}^{\pi} d\phi_i \right) e^{-\beta S} \equiv \int \mathscr{D}\phi\, e^{-\beta S} \qquad \text{(II.24)}$$

as usual. It is also useful to consider the expectation value of

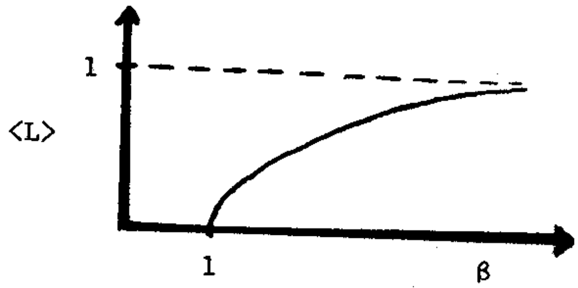
$$L(\vec{r}) \equiv \prod_{n=1}^{N} U_{n,t} \ . \qquad \text{(II.25)}$$

The field $L(\vec{r})$ is just the product of N links in the time direction along a straight line. With periodic boundary conditions, it can be considered a "loop around the universe". It can also be shown (Kuti, et al.;[8] McLerran and Svetitsky[9]; Weiss[10]) that the expectation value of L is related to the free energy of an isolated charge F_e by

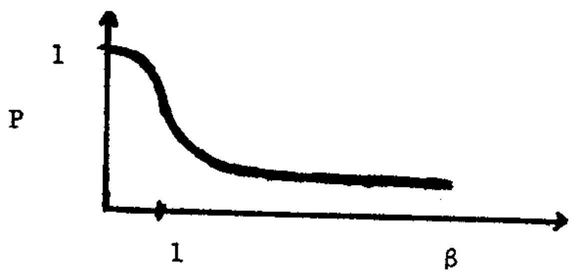
$$\langle L \rangle \equiv \frac{1}{Z} \int \mathcal{D}\phi \; L \; e^{-S}$$
$$= e^{-N^d F_e} \; . \qquad \text{(II.26)}$$

The expectation value ⟨L⟩ can thus be used as a test for confinement in a gauge theory. If changes are unconfined, F_e is finite and so therefore ⟨L⟩ is nonzero. If however charges are confined there are no free charges and F_e is infinite, so ⟨L⟩ = 0. Deconfinement phase transitions are thus signalled by the nonvanishing expectation value of ⟨L⟩.

In two and three dimensions there is no deconfining transition in compact electrodynamics. However, in four dimensions such a transition is present in an infinite lattice system, and a plot of ⟨L⟩ versus β looks like:



The phase transition at β = 1 is second order, so the derivative of the average plaquette is also infinite at that point:



C. Non-Abelian Lattice Gauge Theories

It is not difficult to extend this formalism to non-Abelian gauge theories. For SU(2) in the fundamental representation the $U_{n,\mu}$ are given by,

$$U_{n,\mu} = \exp[iB_\mu(n)] \quad , \tag{II.27}$$

where

$$B_\mu(n) = ag\,\sigma_i A^i_\mu(n) \quad , \tag{II.28}$$

and the σ^i are the familiar Pauli spin matrices. The link $U_{n,\mu}$ can also be written in the form

$$U_{n,\mu} = a_0 I + i\vec{a}\cdot\vec{\sigma} \quad , \tag{II.29}$$

where a_μ is a real Euclidean four-vector of unit length,

$$a_0^2 + \vec{a}^2 = 1 \ . \tag{II.30}$$

In terms of these variables, the invariant group measure takes the form (Creutz[11])

$$dU = \frac{1}{2\pi^2}\,\delta(a^2 - 1)d^4a \tag{II.31}$$

while the action is given by

$$S = \sum \left[1 - \frac{1}{2}\, \mathrm{Tr}(U_{n,\mu} U_{n+\mu,\nu} U^{-1}_{n+\nu,\mu} U^{-1}_{n,\nu})\right] \qquad (II.32)$$

where the sum extends over all plaquettes as usual. The formalism is similar for lattice SU(3) [compact quantum chromodynamics].

In four dimensions there is no deconfining transition for SU(2) or SU(3) in the fundamental representation--$\langle L\rangle$ is zero for all β. It is possible to calculate a quantity known as the string tension in the lattice theory, which is related to the potential energy between two charges [for SU(3) we can call them quarks] in the fundamental representation. This calculation begins with a study of Wilson loops.

For a closed contour C comprised of links in the lattice, the Wilson loop is defined by

$$W(C) = \langle \frac{1}{2}\, \mathrm{Tr}(\prod_C U)_{PO} \rangle , \qquad (II.33)$$

where the subscript PO means "path ordering"; the $U_{n,\mu}$ are ordered and oriented as they are encountered in circulating around the contour. For a confining theory the string tension K is determined from W(C) by

$$\ln W(C) \cong -a^2 K N_\square (C) \qquad (II.34)$$

in the limit of large loop size. Here a is the lattice spacing and $N_\square(C)$ is the number of plaquettes enclosed by the loop. The static potential between two fundamental representation charges is then

$$V(r) \cong Kr , \qquad (II.35)$$

at large distances. Note that if K is greater than zero then it is impossible to separate charges by infinite distance without using infinite energy. Therefore, if K is positive in the continuum limit, then quarks are confined.

A digression on the continuum limit of a lattice gauge theory is relevant here. Not all values of the bare lattice coupling constant g_0 can be identified with a continuum limit. A necessary condition for a continuum theory to exist is that a second order phase transition occur at the value of g_0 under consideration. This condition is mandated by the requirement that the lattice correlation length (in units of the lattice spacing) approach infinity or, alternatively, that the lattice spacing approach zero as the continuum limit is taken. A continuum limit of SU(2) and SU(3) is believed to exist in the limit $\beta \to \infty$ ($g_0 \to 0$).

III. INTERACTION OF GAUGE THEORIES WITH SCALAR MATTER FIELDS

A. Spontaneous Symmetry Breaking and the Higgs Mechanism

The first successful application of non-Abelian gauge theories to particle physics occurred in the development of the theory of the weak interaction (See, e.g., Taylor[3] for a chronology). Let us try to recreate some of the logic which led to such an application.

The so-called "weak" interactions are characterized by various definite signatures (such as parity violation) and by an unusually slow rate of interaction in low energy processes. Cross sections σ are typically proportional to the Fermi weak coupling constant G_F,

$$\sigma \sim G_F \sim \frac{10^{-5}}{M_p^2} , \qquad \text{(III.1)}$$

where M_p is the mass of the proton. Such a small reaction cross section is indicative of a very short-ranged interaction. By the correspondence principle a short-ranged interaction (of range r) can be associated with the exchange of a very massive particle [of mass $M \sim \hbar/(cr)$]. The weak interaction is generally associated with the interchange of particles of roughly 100 times the mass of the proton.

The next step is to construct a consistent quantum field theory involving the exchange of a heavy particle. For reasons which will not be discussed here, a theory of the weak interaction should associate a vector field with the particle. It is aesthetically attractive to model the weak interaction with a local gauge theory. However, a major difficulty remains. The pure gauge theories discussed in the previous section described massless particles. How can we make a local gauge theory of massive particles?

To appreciate the difficulty involved in constructing such a theory, let us see why a simple attempt to do so fails. Remember that a massless scalar field theory is described by a Lagrangian density

$$\mathscr{L}_{\text{massless}} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi^*) \qquad \text{(III.2)}$$

with an Euler equation of motion,

$$\Box \phi = 0 \; . \qquad \text{(III.3)}$$

A massive scalar field is described by essentially the same Lagrangian density, but with a "bare mass" term added:

$$\mathscr{L}_{\text{massive}} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi^*) - \frac{1}{2} m^2 \phi \phi^* \; . \qquad \text{(III.4)}$$

The equation of motion for this massive scalar theory is

$$(\Box + m^2)\phi = 0 \; . \qquad \text{(III.5)}$$

Similarly, the Lagrangian density for, e.g., free U(1) gauge theory is given by

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} ,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \qquad \text{(III.6)}$$

which (in the Lorentz gauge $\partial_\mu A^\mu = 0$) leads to the equation of motion

$$\Box A_\mu = 0 . \qquad \text{(III.7)}$$

The most obvious way of constructing the Lagrangian density for a massive "U(1)" theory would be to add a quadratic mass term,

$$\mathscr{L}_{massive} \overset{?}{=} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu . \qquad \text{(III.8)}$$

This Lagrangian density gives rise to the equation of motion of a massive vector field,

$$(\Box + m^2) A^\mu = 0 , \qquad \text{(III.9)}$$

Unfortunately the Lagrangian density Eq.(III.8) is not invariant under local gauge transformations. It is this lack of local gauge invariance in theories with bare mass terms which leads to inconsistent theories and is the source of the trouble alluded to above.

In order to create a local gauge theory of the weak interactions, it is necessary to "smuggle in" an effective mass term. One way to accomplish this covert operation is by way of a concept known as spontaneous symmetry breaking. Following Abers and Lee[2] , this concept is explored below.

A scalar field theory is very similar to a collection of anharmonic oscillators. Consider a Lagrangian density representing a single-component self-interacting scalar field,

$$\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) + U(\phi^2) , \tag{III.10}$$

with

$$U(\phi^2) = \lambda(\phi^2 - f)^2 . \tag{III.11}$$

For simplicity, let there be only one space dimension. Then the Lagrangian is

$$\begin{aligned} L &= \int_{-\infty}^{\infty} \mathscr{L}(x,t)\, dx \\ &= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U(\phi^2)\right] . \end{aligned} \tag{III.12}$$

Divide space into unit cells of length ε labelled by the coordinate x_i, so that $x_{i+1} - x_i = \varepsilon$. If a separate canonical coordinate $q_i(t) = \phi_i(x_i,t)$ is associated with each cell, the Lagrangian can be written as

$$L = \sum_{i=-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{dq_i}{dt}\right)^2 - \frac{1}{2\varepsilon^2}(q_i - q_{i-1})^2 - U(q_i^2)\right] . \tag{III.13}$$

The canonical momenta are

$$p_i = \frac{dq_i}{dt} , \tag{III.14}$$

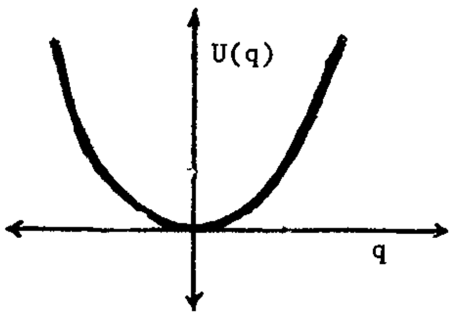
and so the Hamiltonian is given by

$$H = \sum_{i=-\infty}^{\infty} [\frac{1}{2} p_i^2 + V(q_i, q_{i-1})] \quad , \tag{III.15}$$

where

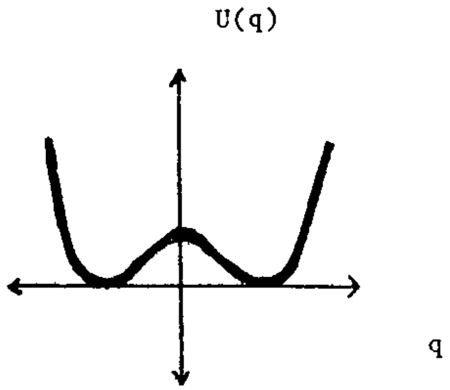
$$V(q_i, q_{i-1}) = \frac{1}{2\varepsilon^2} (q_i - q_{i-1})^2 + U(q_i^2) \quad . \tag{III.16}$$

The Hamiltonian Eq.(III.15) describes a system of anharmonic oscillators. The oscillations are bounded if $\lambda > 0$, which is therefore assumed. In order to ascertain the ground state of the theory, it is necessary to determine the minima of the potential V. At the minima, the q_i are all equal, so that $(q_i - q_{i-1})^2 = 0$. Thus the minima of V are determined by the minima of U. If f is negative, the potential U looks like



so the minimum of U (and thus the ground state of theory) is given by $q_i = 0$. The Lagrangian of the theory is symmetric under the interchange $q_i \to -q_i$, and so is this ground state. Spontaneous symmetry breaking does not occur.

On the other hand, if f is positive, the potential function U(q) looks like,



and so the symmetric minima are at $q_i = \pm\sqrt{f}$. Note that the ground state must have <u>either</u> $q_i = +\sqrt{f}$ <u>or</u> $q_i = -\sqrt{f}$. By convention, the plus sign is chosen. In the original scalar field theory, the vacuum expectation of ϕ is given by $\langle\phi\rangle = \upsilon \cong f$. Although the Lagrangian of the theory is symmetric under the reflection $\phi \to -\phi$, the ground state is not (since $\sqrt{f} \neq -\sqrt{f}$). In this way the reflection symmetry is "spontaneously" broken.

This phenomena may also occur when gauge fields are present. If the scalar field theory discussed above is generalized to a two-component (i.e., complex) theory and coupled to a U(1) gauge field, the Lagrangian is

$$[(\partial_\mu + ieA_\mu)\phi^*(\partial^\mu - ieA^\mu)\phi] - \lambda[(\phi^*\phi) - f]^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \quad . \qquad \text{(III.16)}$$

This Lagrangian is invariant under the local U(1) gauge transformations,

$$\phi(x) \rightarrow \phi(x)\exp[-i\theta(x)] \ ,$$

$$\phi^*(x) \rightarrow \phi^*(x)\exp[+i\theta(x)] \ , \qquad \text{(III.17)}$$

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x) \ ,$$

and corresponds to a self-interacting charged scalar particle.

As before, if f is negative, $\langle\phi\rangle = 0$, and symmetry breaking need not occur. If f is positive, spontaneous symmetry breaking may occur, and in that case $\langle\phi\rangle = \upsilon \cong \sqrt{f}$. In order to define perturbation theory about this vacuum it is necessary to define the theory in terms of fields whose vacuum expectation values vanish. The new real fields η and ξ are defined by

$$\phi = \upsilon + \eta + i\xi \ , \qquad \text{(III.18)}$$

where $\langle\eta\rangle = \langle\xi\rangle = 0$.

Written in terms of these new fields, the Lagrangian is

$$\mathscr{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{2}m^2A'_\mu A'^\mu + \ldots \qquad \text{(III.19)}$$

where $A'_\mu = A_\mu - \frac{1}{e\upsilon}\partial_\mu\xi$, $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$, $m^2 = e^2\upsilon^2$, and the ellipsis (...) refers to terms which are higher order in the fields. Although the gauge symmetry of the Lagrangian is no longer manifest, it does have the form of a Lagrangian for a massive vector field A'_μ coupled to a real scalar field* η.

*The alert reader may notice that the field ξ is not present in the new Lagrangian. This disappearance is a standard feature of the Higgs mechanism, and is discussed in Abers and Lee.

This method of "generating" masses for a gauge theory via the Higgs mechanism was successfully applied to a mixture of SU(2) and U(2) gauge theories to produce a theory of the weak interaction (see, e.g., Taylor,[3] Abers and Lee,[2] and the original work by Weinberg[12] and by Salam[13]). Thus the concept of spontaneous symmetry breaking may be relevant in particle physics.

B. The Abelian Higgs Model

As we have seen in the last section, the presence of matter fields can exert a dramatic effect on the behavior of a gauge theory. In the case of the Higgs mechanism, the interaction of the scalar fields with the gauge bosons rendered them massive and thus the forces they mediate short ranged. The importance of this mechanism thus suggests that we study it further.

The Abelian Higgs model provides us with an excellent prototype for the study of more complex models of matter interacting with gauge fields. It consists of a scalar field of fixed magnitude coupled to the field of a massless photon in a U(1) gauge-invariant fashion. As is shown below, the model can be thought of as an appropriate large—self-coupling limit of a lattice version of scalar electrodynamics. The analysis of the Abelian Higgs (fixed magnitude) model may thus furnish insight into the nature of the more general but noncompact (variable magnitude) model. Excellent discussions of the model appear in, e.g., Fradkin and Shenker[14] and references contained therein. Preliminary Monte Carlo results appear in Callaway and Carson;[15] in Bowler, _et al._;[16] and in Ranft, _et al._[17]

The Abelian Higgs model is discussed here on a hypercubic Euclidean lattice with lattice spacing a. A Higgs field $\sqrt{\beta_H}\,\phi_n$ is defined on each site n of the lattice, subject to the constraint

$$|\phi_n|^2 = 1 \quad \text{for all n .} \tag{III.20}$$

Electromagnetic field variables $U_{n\mu}$ are associated with each link connecting site n with site n+μ. The internal symmetry group is U(1), so that ϕ_n and $U_{n\mu}$ are simple phases $e^{i\theta}$, with $-\pi \leq \theta \leq \pi$. The dynamics of the interacting system are governed by the action

$$A = \beta \sum_{\square} A\ (n,\mu,\nu) + \beta_H \sum_{n,\mu} A_H(n,\mu) \tag{III.21}$$

where the plaquette A and scalar A_H actions are given by

$$A\ (n,\mu,\nu) = \left(1 - U_{n,\nu}\ U_{n+\mu,\nu}\ U^{-1}_{n+\nu,\mu}\ U^{-1}_{n,\nu}\right) \ ,$$

$$A_H(n,\mu) = \frac{1}{2} \left|\left(U_{n,\mu}\right)^q \phi_{n+\mu} - \phi_n\right|^2 \ . \tag{III.22}$$

As seen below, the power q is to be identified with the charge of the Higgs field. The action is locally gauge invariant with respect to the set of transformations

$$\left(U_{n,\mu}\right)^q \rightarrow V^{+}_{n+\mu}\left(U_{n,\mu}\right)^q V_n \ ,$$

$$\phi_n \rightarrow V_n \phi_n \ , \tag{III.23}$$

where the $V_n = e^{i\lambda_n}$ are elements of the underlying U(1) gauge group.

The constraint $|\phi_n|^2 = 1$ can also be implemented by allowing the magnitude of the scalar field to vary (i.e., by integrating over this magnitude) but including a suitable potential term in the action. One possible extension of Eq.(III.21) is

$$A' = \beta \sum_{\Box} A \quad (n,\mu,\nu) + \beta_H \sum_{n\mu} A_H(n,\mu) + \lambda \sum_n \left(|\phi_n|^2 - 1\right)^2 . \qquad \text{(III.24)}$$

Note that in the limit $\lambda \to \infty$ configurations in which $|\phi_n|$ is different from unity are not important.

With the interpretation

$$\beta = \frac{1}{e^2} ,$$

$$\sqrt{\beta_H}\ \phi_n = a\Phi(x_n) , \qquad \text{(III.25)}$$

$$U_{n,\mu} = \exp[iaeA_\mu(x_n)] ,$$

in terms of the lattice spacing a and the electric charge e, the naive continuum limit ($a \to 0$) of Eq.(III.24) yields the familiar Euclidean action of scalar electrodynamics:

$$A'_{\text{continuum}} = \int d^4x\Big[\frac{1}{4} F^2_{\mu\nu}(x) + \frac{1}{2} \left|\left(\partial_\mu + ieqA_\mu\right)\Phi(x)\right|^2$$

$$+ \lambda'\left(|\Phi(x)|^2 - f\right)^2\Big] , \qquad \text{(III.26)}$$

with

$$\lambda' \equiv \frac{\lambda}{\beta_H^2} \qquad \text{(III.27)}$$

and

$$f = \frac{\beta_H}{a^2} . \qquad \text{(III.28)}$$

The parameter q specifies the charge of the Higgs field Φ in units of e.*

Let us return now to the lattice Abelian Higgs model and study its phase diagram in some limiting case:

1) First for $\beta_H = 0$ the action Eq.(III.21) reduces to the action for pure QED, discussed in Section II. Recall that there was one (second order) phase transition as a function of β (at $\beta = 1$).

2) In the limit $\beta \to \infty$ all the $U_{n\mu}$'s are gauge equivalent to unity. Thus the action can be written in the form

$$A \to \beta_H \sum_{n,\mu} [1 - \cos(\theta_n - \theta_{n+\mu})] , \qquad \text{(III.29)}$$

which is the action ("Hamiltonian") for the XY model. In four dimensions, this model has a phase transition at $\beta_H \cong 0.453$.

3) As β_H increases without bound, the action approaches that of a Z_q gauge model, viz:

$$A \to \beta \sum (1 - UUU^{-1}U^{-1}) , \qquad \text{(III.30)}$$

subject to the constraint

*Note that this is merely the naive continuum limit of the theory. Extracting the continuum limit in this fashion, strictly speaking, is invalid since the simple $a \to 0$ limit is singular. See Callaway and Carson.15

$$A_H = 0 \ . \tag{III.31}$$

In the unitary gauge ($\phi_n = 1$), the constraint Eq.(III.31) can be written

$$\sum_{n\mu} [1 - \mathrm{Re}(U_{n\mu})^q] = 0 \ , \tag{III.32}$$

or

$$(U_{nq})^q = 1 \ . \tag{III.33}$$

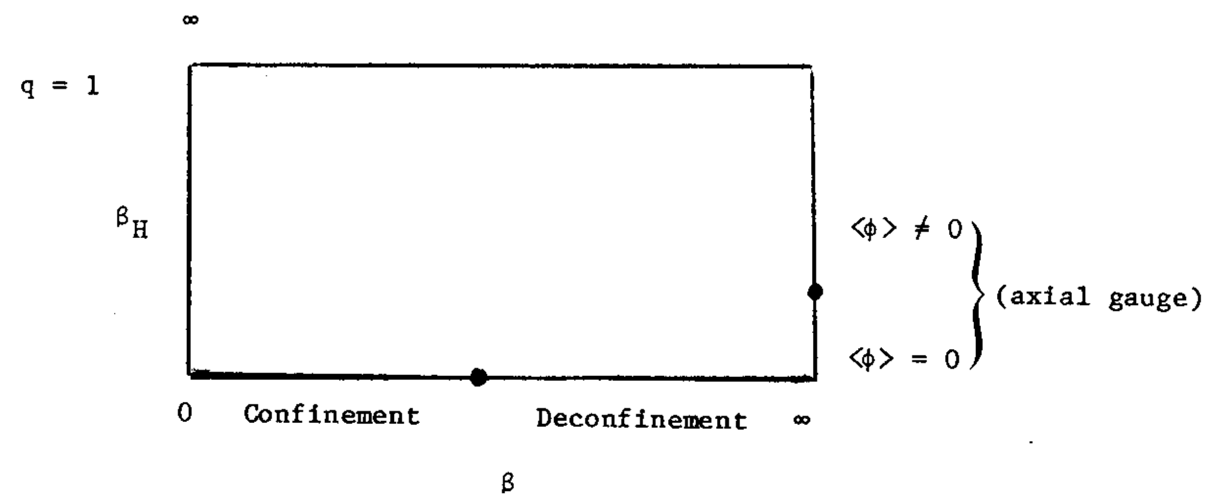
In this discussion we will look at the q = 1 and q = 2 Higgs phase diagrams. For q = 1, the constraint Eq.(III.33) simply implies that all the U's are gauge-equivalent to unity, and so the theory is trivial in the limit of large β_H. For q = 2, this constraint restricts the variables $U_{n\mu}$ to be equal to ±1 (in the unitary gauge); the theory thus approaches a Z_2 gauge model in this limit. The four-dimensional Z_2 lattice gauge theory has a phase transition at $\beta_H = \frac{1}{2} \ln(1 + \sqrt{2}) \cong 0.44069$.

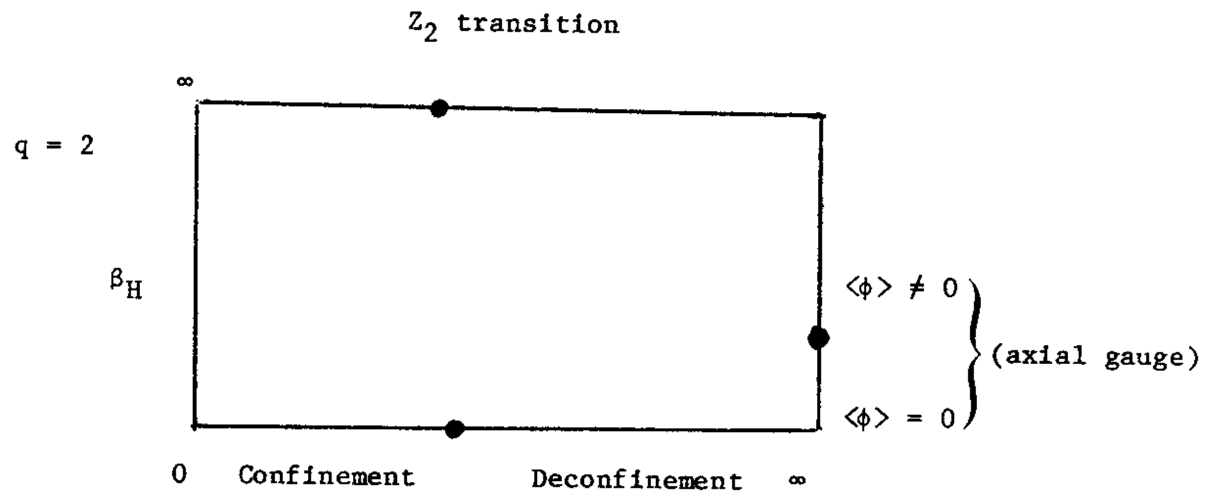
4) For $\beta = 0$ the action in the unitary gauge takes the form

$$A = \beta_H \sum_{n,\mu} [1 - \mathrm{Re}(U_{n,\mu})^q] \ . \tag{III.34}$$

The partition function factorizes in this limit, and so the model is trivial (no phase transition occurs).

These results can be used to sketch in the phase diagrams of the q = 1 and q = 2 theory near their edges:





The above analysis suggests the existence of (at least) three distinct phases in the Abelian Higgs model. Properties of these phases are listed below:

1) Higgs Phase:

Massive gauge bosons ($\langle\phi\rangle \neq 0$)

Short range forces between charges

2) Coulomb (or Electrodynamics) Phase:

Massless gauge bosons ($\langle\phi\rangle = 0$)

Coulomb force law between charges

Free charges exist (F_e finite, $\langle L\rangle \neq 0$)

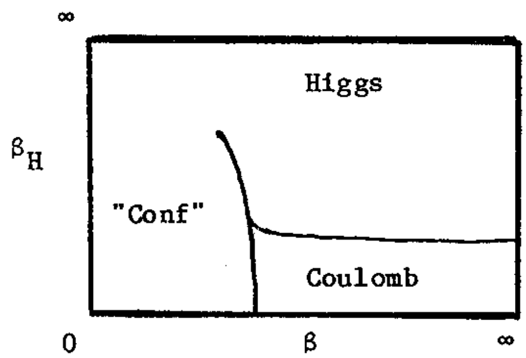
3) Confinement Phase

Massive gauge bosons

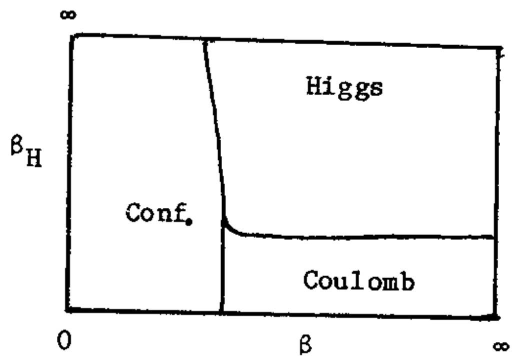
No free charges ($F_e = \infty$, $\langle L\rangle = 0$)

The full phase diagram for q = 1 and 2 has been determined by Monte Carlo methods (Callaway and Carson;[15] Bowler et al.;[16] Ranft, et al.[17]). With the aid of the above discussion, the various phases may be identified:

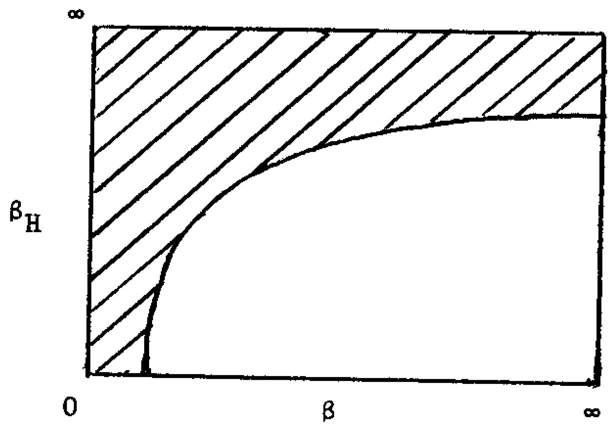
q = 1



$q = 2$



Notice a rather surprising result--for $q = 1$ the Higgs and confinement phases are continuously connected. In fact, it can be shown (Fradkin and Shenker[14]) that there is a region of analyticity for $q = 1$

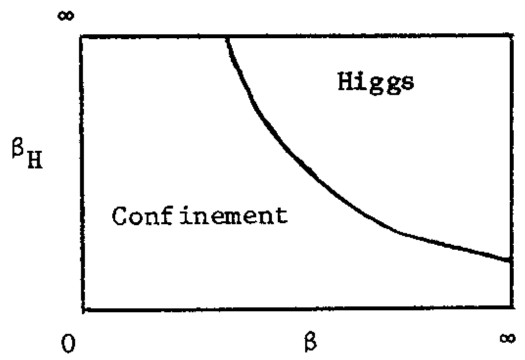


in the (β, β_H) plane where the free energy and its derivatives are analytic. The physical origin of this analyticity arises from the fact that charge one Higgs can screen fundamental representation ($q = 1$) fermions and thus spoil confinement. On the other hand, a charge two Higgs particle can never screen a charge one fermion, and so confinement is absolute for $q = 2$.

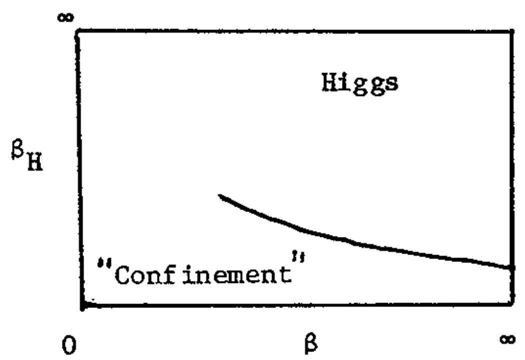
C. Nonabelian Higgs Models

The above analysis has been generalized to the case of an SU(2) lattice gauge theory interacting with fundamental representation Higgs fields [corresponding to q = 1 in the U(1) case], and adjoint representation Higgs fields [corresponding to q = 2]. The phase diagrams were analyzed by Brower, *et al.*[18] and Lang, *et al.*,[19] and are sketched below:

FUNDAMENTAL REPRESENTATION HIGGS:



ADJOINT REPRESENTATION HIGGS



Note that, except for the lack of a deconfining transition for $\beta_H = 0$, the phase diagrams are remarkably similar to those for the Abelian Higgs model. In both cases, fundamental representation scalars can screen fundamental representation fermions and spoil confinement. Thus the confinement and Higgs phases are connected when the Higgs fields are in a fundamental representation.

IV. FERMIONS ON A LATTICE

A. Basic Formalism

In order to complete a lattice version of the theory of the strong interaction, we have to have a way of putting quarks on the lattice. Quarks are fermions, and their inclusion in lattice gauge theories has been the source of most of the frustrations of the field. Their addition to the lattice framework requires the introduction of a strange algebra--Grassmann variables--and their simulation by numerical means is guaranteed to strain the capacity of the most advanced computing systems. Yet with all their difficulty they are essential, for they are the fundamental components of hadrons.

In the path integral formulation of quantum field theory, fermions are represented by Grassmann variables. A detailed discussion of these variables can be found in Berezin[20] (see also Kadanoff;[21] Wilson[22]), however, a few simple preperties of this algebra are all that is required here. The basic elements of this algebra are statistical variables Ψ_i, $\bar{\Psi}_i$ which anticommute:

$$\Psi_i\Psi_j + \Psi_j\Psi_i \equiv \{\Psi_i,\Psi_j\} = 0$$

$$\bar{\Psi}_i\bar{\Psi}_j + \bar{\Psi}_j\bar{\Psi}_i = \{\bar{\Psi}_i,\Psi_j\} = 0 \ . \tag{IV.1}$$

These relations hold for all i and j, and specifically for i = j. Thus the square of any element is zero:

$$(\Psi_i)^2 = (\bar{\Psi}_i)^2 = 0 \ . \tag{IV.2}$$

The relations Eqs.(IV.1) and (IV.2) imply that any function of the Grassmann variable Ψ is of the form

$$f(x) = a + b\Psi \tag{IV.3}$$

where a and b are real numbers. In order to perform a path integration a table of integrals for Grassmann variables is needed. The first entry in our table is the integral

$$\int d\Psi \; \Psi = 1 \; , \tag{IV.4}$$

where the value of unity is chosen by convention. The value of the integral is not affected by the translation $\Psi \to \Psi + \eta$, where η is another Grassmann variable. Thus

$$\begin{aligned} \int d\Psi \; \Psi &= \int d(\Psi + \eta)(\Psi + \eta) \\ &= \int d\Psi \; \Psi + \int d\Psi \; \eta \\ 1 &= 1 - \eta \int d\Psi \; . \end{aligned} \tag{IV.5}$$

Therefore, the next integral in our table is

$$\int d\Psi = 0 \; . \tag{IV.6}$$

All other integrals over a Grassmann algebra can be written as linear combinations of the integrals Eqs.(IV.4) and (IV.6). Thus our table of Grassmann integrals is complete.

Notice that the operation of integration over a Grassmann algebra cannot be represented as a sum with positive weights. For this reason some of the standard theorems of statistical mechanics fail for systems represented with Grassmann variables.

Before proceeding further, let us calculate a simple partition function involving two Grassmann variables in order to illustrate the above concepts. The action of our system is taken to be

$$S = A \bar{\Psi} \Psi , \tag{IV.7}$$

and so the partition function is

$$Z = \int d\Psi \int d\bar{\Psi} \exp(-A \bar{\Psi} \Psi)$$

$$= \int d\Psi \int d\bar{\Psi} (1 - A \bar{\Psi} \Psi) \tag{IV.8a}$$

(since $\Psi^2 = \bar{\Psi}^2 = 0$)

$$= -A \int d\Psi \left(\int d\bar{\Psi} \bar{\Psi}\right) \Psi \tag{IV.8b}$$

(since $\int d\bar{\Psi} = 0$)

$$= -A \tag{IV.8c}$$

(since $\int d\Psi \Psi = \int d\bar{\Psi} \bar{\Psi} = 1$).

Note that if A is positive, the partition function is strictly negative. Certainly we are in the midst of strange concepts.

Another peculiar result arises if a multiplicative change of variables $\Psi \to \lambda\Psi \equiv \Psi'$ is made. Since the value of the integral must remain fixed under this change of variables,

$$\begin{aligned} \int d\Psi\ \Psi &= \int d\Psi'\ \Psi' \\ &= \int d\Psi'\ \lambda\Psi \end{aligned} \tag{IV.9}$$

it follows that if

$$\Psi' = \lambda\Psi \tag{IV.10a}$$

then

$$d\Psi' = \frac{1}{\lambda}\ d\Psi \tag{IV.10b}$$

in distinct contrast to the laws of integration we are used to.

B. Fermions Without Gauge Fields

The above analysis is now generalized to a system with many degrees of freedom. A further discussion can be found in Kadanoff.[21] Consider a fermionic system with a single fermion action

$$S(1) = -\ \bar{\Psi}(1)\Psi(1) + \bar{\Psi}(1)\Sigma(1,\bar{2})\Psi(\bar{2}) + \bar{\Psi}(\bar{2})\Sigma(\bar{2},1)\Psi(1)\ . \tag{IV.11}$$

The action for the system is the sum of this single coordinate action over all fields 1. We adopt the convention that all barred indexes are summed over. Note that $\Psi(1)$ is a vector and $\Sigma(1,2)$ a matrix. For convenience the notation

$$\eta(1) \equiv \Sigma(1,\bar{2})\Psi(\bar{2}) \ ,$$

$$\bar{\eta}(1) \equiv \bar{\Psi}(\bar{2})\Sigma(\bar{2},1) \ , \qquad \text{(IV.12)}$$

$$\mathrm{Tr}_1 \equiv \int d\Psi_1 \int d\bar{\Psi}_1 \ ,$$

is used.

Because the Grassmann variables anticommute, it follows that

$$e^{S(1)} = 1 - \bar{\Psi}(1)\Psi(1) + \bar{\Psi}(1)\eta(1) + \bar{\eta}(1)\Psi(1) - \bar{\Psi}(1)\Psi(1)\bar{\eta}(1)\eta(1) \ . \qquad \text{(IV.13)}$$

The two-point Green's function $G(1,2)$ is defined by

$$G(1,2) \equiv \frac{1}{Z}\,[\prod_n \mathrm{Tr}_n e^{S(n)}]\Psi(1)\bar{\Psi}(2) \ , \qquad \text{(IV.14a)}$$

with

$$Z = [\prod_n \mathrm{Tr}_n e^{S(n)}] \ . \qquad \text{(IV.14b)}$$

It is easy to see that

$$Tr_1 e^{S(1)} = -1 + \eta(1)\bar{\eta}(1) \tag{IV.15a}$$

$$Tr_1 e^{S(1)} \Psi(1)\bar{\Psi}(2) = -\eta(2)\bar{\Psi}(2) \quad (1 \neq 2) \tag{IV.15b}$$

$$= -1 \quad (1 = 2) \tag{IV.15c}$$

and therefore

$$Tr_1 e^{S(1)} \Psi(1)\bar{\Psi}(2) = Tr_1 e^{S(1)} \eta(2)\bar{\Psi}(2) \quad (1 \neq 2) \tag{IV.16a}$$

$$= Tr_1 e^{S(1)}[1 + \eta(1)\bar{\eta}(1)] \quad (1 = 2) . \tag{IV.16b}$$

The Green's function can then be written

$$\begin{aligned} G(1,2) &= [1 - \delta(1,2)]\Sigma(1,\bar{1})G(\bar{1},2) \\ &+ \delta(1,2)[1 + \Sigma(1,\bar{1})G(\bar{1},\bar{2})\Sigma(\bar{2},2)] \end{aligned} \tag{IV.17}$$

where $\delta(1,2)$ is the Kronecker delta symbol. After some symbol manipulation we obtain

$$G(1,2) = \Sigma(1,\bar{1})G(\bar{1},2) + \delta(1,2) \tag{IV.18}$$

which has the symbolic solution

$$G = \frac{1}{1 - \Sigma} . \tag{IV.19}$$

The partition function is also easy to evaluate. Note that if we change Σ infinitesimally,

$$\delta(\ell n\ Z) = \frac{1}{Z}\left[\prod_n \mathrm{Tr}_n\ e^{S(n)}\right][\delta\Sigma(\bar{1},\bar{2})]\bar{\Psi}(\bar{1})\Psi(\bar{2})$$

$$= -[\delta\Sigma(\bar{1},\bar{2})]G(\bar{2},\bar{1})\ , \qquad \text{(IV.20)}$$

where the minus sign arises because we must anticommute Ψ and $\bar{\Psi}$. This relation can be integrated to read

$$\delta\ (\ell n\ Z) = \delta\ [\mathrm{Tr}\ \ell n(1 - \Sigma)] \qquad \text{(IV.21)}$$

and so, since $Z = 1$ if $\Sigma = 0$,

$$\ell n\ Z = \mathrm{Tr}\ \ell n(1 - \Sigma) \qquad \text{(IV.22)}$$

or

$$Z_{\mathrm{fermion}} = \exp\ \mathrm{Tr}\ \ell n(1 - \Sigma)$$

$$= \det\ (1 - \Sigma)\ . \qquad \text{(IV.23)}$$

Thus, to calculate the fermionic partition function we need the determinant of the matrix $1 - \Sigma$. Parenthetically, it is interesting to note that the corresponding result for scalar (boson) fields is [compare Eqs.(IV.10)]

$$\ell n\ Z_{boson} = -\mathrm{Tr}(1 - \Sigma) ,$$

or (IV.24)

$$Z_{boson} = [\det(1 - \Sigma)]^{-1} .$$

Now that we have developed a formalism, let's apply it to the problem of calculating the free fermion propagator on a lattice. This propagator has internal spinor indices which appear in multiplication of the Dirac matrices γ_α, where α ranges from one to four. The Dirac matrices obey the Euclidean version of the standard anticommutation relations, i.e.,

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta(\alpha, \beta) . \tag{IV.25}$$

In the Wilson formulation of lattice gauge theory, the action for bare fermions has the form

$$S = K \sum_{n,\mu} [\bar{\Psi}(n)(1 - \gamma_\mu)\Psi(n - \mu) + \bar{\Psi}(n)(1 + \gamma_\mu)\Psi(n + \mu)] - \sum_n \bar{\Psi}(n)\Psi(n) , \tag{IV.26}$$

(with $\bar{\Psi} = \Psi^+\gamma_0$) which gives

$$\Sigma(x_1, x_2) = K \sum_\mu [(1 - \gamma_\alpha)\delta(x_1 - x_2 - \hat{e}_\alpha a) + (1 + \gamma_\alpha)\delta(x_1 - x_2 + \hat{e}_\alpha a)] , \tag{IV.27}$$

so that

$$\Sigma(p) = 2K \sum_\mu \left[\cos(p_\mu a) - i\gamma_\mu \sin(p_\mu a)\right] ,$$

$$G(p) = \frac{1}{1 - \Sigma(p)} . \qquad \text{(IV.28)}$$

Next define a new variable m,

$$m = \frac{d}{a}(1 - 2dK) , \qquad \text{(IV.29)}$$

where d is the dimensionality of the lattice system. If in taking the continuum limit $(a \to 0)$ K is allowed to approach 1/2d in such a fashion that m remains finite, then we recover the continuum Euclidean Dirac propagator for a fermion of mass m;

$$G_{\text{continuum}}(p) = \frac{1}{m - i\gamma_\mu p^\mu} = \frac{m + i\gamma_\mu p^\mu}{p^2 + m^2} . \qquad \text{(IV.30)}$$

C. Fermions Interacting with Gauge Fields

It is easy to see how to include the interaction of gauge fields with fermions in the Wilson formalism. In addition to the usual pure gauge part of the action, there is an interaction term

$$S_F \equiv K \sum_{n,\mu} \left[\bar{\Psi}(n)U^{+}_{n,\mu}(1 - \gamma_\mu)\Psi(n - \mu) + \bar{\Psi}(n)(1 + \gamma_\mu)U_{n,\mu}\Psi(n+\mu)\right]$$

$$- \sum_n \bar{\Psi}(n)\Psi(n) . \qquad \text{(IV.31)}$$

Equation (IV.31) is, of course, just the gauge-invariant generalization of the pure fermionic action Eq.(IV.26). If V_n is an element of a unitary gauge group of which $U_{n,\mu}$ is a member, then S_F is invariant under the local gauge transformation

$$\Psi_n \to V_n \Psi_n$$

$$\bar{\Psi}_n \to \bar{\Psi}_n V_n^{+} \qquad \text{(IV.32)}$$

$$U_{n,\mu} \to V_n U_{n,\mu} V_{n+\mu}^{+} .$$

Integrations are performed over all fermion fields (represented by the Grassmann algebra) and all gauge fields. Thus the partition function is given by

$$Z_{\text{gauge+fermion}} = \int \mathscr{D}U \, \mathscr{D}\Psi \, \mathscr{D}\bar{\Psi} \, \exp[- S_{\text{gauge}} - S_F]$$

$$= \int \mathscr{D}U \, \exp[-S_{\text{gauge}}(U)] \det[1 - \Sigma(U)] , \qquad \text{(IV.33)}$$

where the matrix $\Sigma(U)$ is the gauge-invariant generalization of Eq.(IV.27). In practice, the determinant appearing in Eq.(IV.33) makes the evaluation of expectation values by Monte Carlo techniques very difficult, for it is a highly nonlocal function and requires a relatively large amount of computer time to evaluate.

Expectation values of various fermionic operators can be calculated in the ensemble defined by Eq.(IV.33). Typical examples include the propagators of hadronic states, which have the form

$$G_{had}(x) \sim \langle\bar{\Psi}(0)\Psi(0)\bar{\Psi}(x)\Psi(x)\rangle \qquad (IV.34)$$

for mesonic (two quark) states. In this case, the gauge group is SU(3). On a periodic lattice of length L, the propagator at large distances should approach the form

$$G_{had}(x) \sim e^{-m(L-x)} + e^{-mx} \qquad (IV.35)$$

where m is the ground state mass of the meson under consideration. Thus meson masses can be calculated by evaluating the vacuum expectation value of a product of four operators. Similarly a baryon mass can be extracted from an expectation value of the product of six operators. [See Hamber and Parisi,[23] and Weingarten].[24]

V. SUMMARY AND OUTLOOK

During the course of these lectures, we have encountered many of the concepts presently used in elementary particle theory. We have explored the concepts of local gauge symmetry--one of the building blocks of modern theories of particle interactions--and used it to construct a Lagrangian. From there we developed the concepts which allowed us to formulate a quantum field theory.

True to the title of these lectures these gauge theories were then cast in a discrete form after the fashion of Wilson. With this prototype in hand we then explored the ways of coupling scalar (boson) matter fields to the theory. This led to the application of the concept of spontaneous symmetry breaking to particle theory. One result was the so-called Higgs mechanism, which allowed the generation of massive gauge bosons without violating local gauge invariance. The presence of these scalar matter fields also produced a new phase--the so-called "Higgs phase"--in the phase diagram of the lattice theory.

From there we moved on to discover fermionic matter fields and how they can be brought into a lattice theory. This required the use of a strange new algebra based on Grassmann variables. These variables anticommute with each other and so many of the standard theorems of statistical mechanics (such as the positivity of the partition function) fail when they are present. Thus it is possible however to define fermionic Green's functions and to calculate hadronic properties such as the mass spectrum from them.

These lectures have almost studiously avoided discussing particular techniques for attacking various problems. Rather, the focus here has been on elucidating the formalism and setting up several of the problems which evolve. Various numerical techniques--Monte Carlo and others--for the study of lattice theories exist (see, e.g., Creutz,[25] Parisi and Wu [26], Callaway and Rahman[27] and references contained therein). There are also analytical techniques for this study (see, e.g. Wilson,[22] Kogut,[28] and references contained therein). All of these methods have their successes and failures. Yet through their use, we are finally developing an understanding of one of the most challenging problems in modern physics--the nonperturbative structure of quantum field theory.

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