TRIVIALITY PURSUIT: CAN ELEMENTARY SCALAR PARTICLES EXIST?

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Abstract:

Great effort is presently being expended in the search for elementary scalar "Higgs" particles. These particles have yet to be observed. The primary justification for this search is the theoretically elegant Higgs–Kibble mechanism, in which the interactions of elementary scalars are used to generate gauge boson masses in a quantum field theory. However, strong evidence suggests that at least a pure ϕ^4 scalar field theory is *trivial* or noninteracting. Should this triviality persist in more complicated systems such as the standard model of the weak interaction, the motivation for looking for Higgs particles would be seriously undermined. Alternatively, the presence of gauge and fermion fields can rescue a pure scalar theory from triviality. Phenomenological constraints (such as a bounded or even predictable Higgs mass) may then be implied. In this report the evidence for triviality in various field theories is reviewed, and the implications for high energy physics are discussed.

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1. Introduction and overview

1.1. Prolegomena

The concept of particle mass generation by spontaneous symmetry breaking [1.1–1.11] has become one of the most important ideas in field theory. Most typically this symmetry breaking is produced by the introduction of a scalar field [1.12–1.17]. This new field supposedly interacts with itself (via a ϕ^4 coupling) in such a fashion as to produce a ground state for the theory which lacks a symmetry of the Lagrangian. At the semiclassical level, gauge particles propagating in the asymmetric vacuum appear to be massive, while the underlying theory remains renormalizable [1.18–1.25]. It is generally assumed that no significant changes occur in the full quantum theory due to effects such as nonperturbative renormalization.

Such an assumption may not be correct however. A large body of analytical and numerical evidence (reviewed in detail in sections 2–4) has been presented which suggests strongly that at least *pure* ϕ^4 field theory is "trivial" or noninteracting in four dimensions. Of course, if the theory does not interact, its (nonexistent) interactions cannot break the symmetry of the vacuum, and gauge particles cannot acquire a mass in this fashion. Should this triviality persist in realistic theories such as the standard model of the weak interaction, it is fair to question the whole idea of symmetry breaking by elementary scalars. In this case the standard model might be a low-energy effective model of a more complete theory (and *ipso facto* Higgs particles may not exist); this possibility is considered in further detail in section 5.

A more interesting scenario may occur in realistic theories where [as in the case of the O(N) gauge–Higgs system in the large-N limit] gauge fields can transform a trivial ϕ^4 theory into a nontrivial model. Although the situation with the standard model is not clear, one realistic possibility is that the Higgs mass is bounded or predictable (see section 5).

1.2. Historical perspectives

The concept of triviality had its origins in the physical ideas of Landau, Pomeranchuk and collaborators [1.26–1.40] over thirty years ago. Landau, Abrikosov, and Khalatnikov summed the leading-logarithmic terms for the photon propagator $D^{\mu\nu}(p^2)$ of quantum electrodynamics in the limit of large momentum transfer p^2. Their result, when written in the "Landau" gauge, is

$$\alpha_R D^{\mu\nu}(p) = -\frac{1}{p^2}\left(g^{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) d(p^2, \alpha_R)\,, \tag{1.1}$$

where, in this limit, $d(p^2, \alpha_R)$ has a power series expansion

$$d(p^2, \alpha) = \alpha_R\left[1 + \frac{\alpha_R t}{3\pi} + \left(\frac{\alpha_R t}{3\pi}\right)^2 + \cdots\right] \tag{1.2a}$$

$$= \frac{\alpha_R}{1 - \alpha_R t/3\pi}\,, \tag{1.2b}$$

while

$$t \equiv \ln(p^2/m^2) \tag{1.3}$$

and α_R is the renormalized fine structure constant, with m the electron mass $(p^2 \gg m^2)$.

If the result eq. (1.2b) is taken as exact, the photon propagator has a pole at the momentum scale

$$p^2 = m^2 \exp(3\pi/\alpha_R)\,. \tag{1.4}$$

This pole is of course the famous "Feldman–Landau ghost" [1.26, 1.41] and leads to a number of inconsistencies (e.g. tachyons) in the full theory [1.29, 1.35, 1.36, 1.42] if α_R is nonzero. One might therefore jump to the conclusion that quantum electrodynamics (QED) is "trivial", i.e., that the theory is inconsistent unless α_R vanishes.

This conclusion is not warranted solely on the basis of such flimsy arguments, however, for the resummation eq. (1.2b) is not justified when $\alpha_R t/3\pi \sim 1$. In fact (as is usually mentioned in elementary texts on quantum field theory) the "ghost" pole does not appear for momentum scales less than many orders of magnitude greater than the mass of the entire known universe, so as a practical matter its consequences for QED are nil. Nevertheless, in principle such an inconsistency (if real) is worrisome.

The physical mechanism at work here is a good deal less mysterious—it is simply the phenomenon of *charge screening* [1.29, 1.30, 1.43]. If a bare electric charge is placed in the vacuum, it will create virtual electron–positron pairs. The charges of like sign will be repelled, while those of opposite sign are attracted. Eventually a "cloud" of virtual charge surrounds the bare test charge and reduces the value of the charge seen at large distances. As probes of higher and higher momentum are applied, the test charge is approached more closely and the observed charge seemingly increases. Disaster occurs if the effective charge seen by the probe becomes infinite at a finite momentum scale.

1.3. Triviality and the renormalization group

For a deeper understanding of the nature of such inconsistencies, it is useful to study the problem by means of the renormalization group (see section 3 and refs. [1.44–1.50]). The basic idea of the renormalization group (which here means the renormalization group of Gell-Mann and Low, Callan, Symanzik and others, rather than the more modern version due to Wilson [1.51, 1.52]) is that the momentum scale μ at which the physical coupling constants of a consistent "renormalizable" theory are defined is arbitrary.

Specifically, a renormalized coupling constant λ_R is defined by a measurement to be performed at a certain momentum scale μ. The theory itself must be equally valid for any choice of μ, provided that the physical couplings are appropriately redefined. If μ is rescaled by

$$\mu^2 \to \mu^2\, e^t \equiv [\bar{\mu}(t)]^2\,, \tag{1.5}$$

then a physically measurable quantity $G(\mu)$ in the original theory must equal its value in the rescaled theory, implying an implicit rescaling of λ_R:

$$G[\bar{\lambda}(t),\, \bar{\mu}(t)] = G[\lambda_R,\, \mu]\,. \tag{1.6}$$

The dependence of $\bar{\lambda}(t)$ upon t can be expressed in terms of a "beta function" $\beta[\bar{\lambda}(t)]$,

$$\partial\bar{\lambda}(t)/\partial t = \beta[\bar{\lambda}(t)]\,. \tag{1.7}$$

For quantum electrodynamics, to lowest order in $\bar{\alpha}$ [1.44, 1.45],

$$\beta(\bar{\alpha}) = \frac{\bar{\alpha}^2}{3\pi} + \cdots, \tag{1.8}$$

implying that (for small $\bar{\alpha}$ at least)

$$\bar{\alpha}(t) = \frac{\alpha_R}{1 - \alpha_R t/3\pi}, \tag{1.9}$$

where $\bar{\alpha}(0) \equiv \alpha_R$ fixes the initial condition.

This "running coupling" $\bar{\alpha}(t)$ is equal to the fine structure constant evaluated at squared momenta scaled by e^t compared to the squared standard momenta at which α_R is defined. The problem of the "ghost" has again manifested itself, for the running coupling appears to become infinite at a finite momentum scale. Again, for quantum electrodynamics there may be no difficulties in the full theory, for, when $\bar{\alpha}(t)$ is large, higher-order terms in $\beta(\bar{\alpha})$ become important and the ghost might vanish.

1.4. Ghostbusting: The route to a consistent theory

It would be convenient if the Landau ghost could be exorcised in this fashion by the inclusion of a few more terms in the beta function. Sadly this is not the case. The beta function for QED has been evaluated to order $\bar{\alpha}^4$ [1.53]:

$$\beta_{\mathrm{QED}}(\bar{\alpha}) = \frac{\bar{\alpha}^2}{3\pi} + \frac{\bar{\alpha}^3}{4\pi^2} - \frac{121}{288}\frac{\bar{\alpha}^4}{\pi^3} + \cdots \tag{1.10}$$

and the appropriate finger exercises rapidly reveal the persistence of the ghost within the perturbative domain. To this day it is not known whether QED is fundamentally self-consistent.*)

Physical intuition does not fail us however. This inconsistency is just the problem of charge screening mentioned above. As the charge is redefined at higher and higher momentum scales, it appears to increase without bound. What is needed (heuristically speaking) is a way to ensure that the physical charge at any given realizable finite momentum scale is itself finite. This criterion can be formulated in terms of the beta function of a theory. Consider the solution of eq. (1.7),

$$t = \int_{\lambda_R}^{\bar{\lambda}(t)} \frac{\mathrm{d}\lambda}{\beta(\lambda)}\,. \tag{1.11}$$

The requirement that $\bar{\lambda}(t)$ be bounded for all finite positive t can be satisfied if $\beta(\lambda)$ is continuous and obeys one of the following two conditions:

*) Actually this discussion is a bit naive. It has been argued [1.82, 1.83] that, if a nontrivial fixed point exists in quantum electrodynamics, it must be an infinite-order zero of the beta function. See also ref. [1.84]. Recent calculation suggest that such a fixed point may exist [1.86].

Ghostbusting condition (a). $\bar{\lambda}(t)$ approaches a fixed point λ^* as $t \to +\infty$ such that $(\lambda^* - \bar{\lambda})^{-1}\beta(\bar{\lambda})$ is bounded as $\lambda \to \lambda^*$ [thus $\beta(\lambda^*) = 0$].

If $\beta(\bar{\lambda}) > 0$ for $\lambda < \lambda^*$, $\bar{\lambda}$ approaches λ^* from below. Then by assumption $(\lambda^* - \bar{\lambda})^{-1}\beta(\bar{\lambda})$ is bounded in this limit by some positive constant, let us call it c. It follows that t increases without bound as $\bar{\lambda}(t)$ approaches λ^*, i.e., that if λ_R is finite then $\bar{\lambda}(t)$ is finite for all positive t. From eq. (1.11),

$$t = \int_{\lambda_R}^{\bar{\lambda}} \frac{d\lambda}{\lambda^* - \lambda} \frac{\lambda^* - \lambda}{\beta(\lambda)} \geq -\frac{1}{c} \ln(\lambda^* - \bar{\lambda}) + \text{finite constant},$$

and thus $t \to +\infty$ for finite $\bar{\lambda}$. The demonstration is similar if $\bar{\lambda}$ approaches λ^* from above.

*Ghostbusting condition (b). $\bar{\lambda}(t)$ increases without bound as $t \to \infty$ but $\bar{\lambda}^{-1}\beta(\bar{\lambda})$ is bounded in this limit.**)

The logic inspiring condition (b) is less crepuscular when the beta function $\beta(x)$ for the inverse coupling $x \equiv \lambda^{-\varepsilon}$ (with ε an arbitrary "small" positive constant) is evaluated,

$$\frac{\partial \bar{x}}{\partial t} = \frac{\partial}{\partial t}(\bar{\lambda}^{-\varepsilon}) = -\varepsilon \bar{\lambda}^{-\varepsilon} \frac{\beta(\bar{\lambda})}{\bar{\lambda}} \equiv \tilde{\beta}(\bar{x}). \tag{1.12}$$

It is immediately seen that condition (b) simply means that $\tilde{\beta}(\bar{x})$ vanishes at the fixed point $x^* = 0$, i.e., that an appropriately defined fixed point exists at infinite $\bar{\lambda}$. Taken together conditions (a) and (b) imply that a fixed point λ^* must exist at some value of $\bar{\lambda}(t)$ which is approached as t increases without bound. Conditions (a) and (b) are usually summarized by the statement that an "ultraviolet" fixed point exists for the theory.

It is a common misapprehension that a theory must be "asymptotically free", that is, possess an attractive ultraviolet fixed point at $\lambda^* = 0$, in order to be nontrivial. The preceding discussion shows that this need not be the case. All that is *a priori* necessary is that *some* fixed point exist; thus asymptotic freedom is *sufficient* but evidently unnecessary in a nontrivial theory. However, the demonstration of a nonzero fixed point ($\lambda^* \neq 0$) may well require nonperturbative methods. Some candidate techniques are discussed in the following sections, see especially section 5.

It should be pointed out that the previous discussion is somewhat heuristic. For instance, the concept of a beta function for a trivial theory was used but never explained. A more careful adumbration of these fundamental concepts could begin the analysis by defining a field theory by means of an ultraviolet cutoff. Next, for a given cutoff, the renormalized coupling constant is calculated in terms of the bare coupling. Requirements of consistency (e.g. the existence of a vacuum state in the theory) constrain the bare coupling, and *ipso facto* the renormalized coupling. The ultraviolet cutoff is then removed to infinity; if the limit of these constraints is the requirement that the coupling constant be zero then the theory is a trivial one. This construction forms the implied logical foundation of the preceding discussion. A more modern approach (via the Wilson renormalization group) is given in section 3.

*) This type of behavior can occur in beta functions of supersymmetric Yang–Mills theories, as evaluated by instanton methods. See ref. [1.85].

1.5. Phenomenological implications of triviality in scalar field theories

The topic of interest in this review is of course scalar ϕ^4 field theory, and so without delay we present the result for the lowest-order beta function for a single-component (i.e. $N=1$) ϕ^4 theory,

$$\beta(\bar{\lambda}) = \frac{3}{32\pi^2}\,\bar{\lambda}^2 + \cdots, \tag{1.13}$$

for an interaction Lagrange density

$$L_{\mathrm{I}}(\phi) = -\lambda\phi^4/4!\,. \tag{1.14}$$

(This result can be found in many standard references, e.g. refs. [1.49, 1.54–1.57].) The solution of the renormalization group equations is essentially equivalent to the "parquet sum" of statistical mechanics [1.31, 1.33, 1.58, 1.59] (further references can be found in ref. [1.60]). This lowest-order calculation produces a Landau ghost pole in the running coupling constant $\lambda(t)$,

$$\lambda(t) = \frac{\lambda_{\mathrm{R}}}{1 - 3\lambda_{\mathrm{R}} t/32\pi^2}\,, \tag{1.15}$$

provided that λ_{R} (the renormalized coupling defined at t equal to zero) is positive. This basic result does not change in any lowest-order calculation for multicomponent scalar fields unless non-Abelian gauge fields are included (see section 5 and refs. [1.61–1.63, 1.49]). Additionally in the standard formulation of the theory λ_{R} must be positive if the vacuum energy is bounded from below [1.49, 1.54, 1.55]. (In general, for stability any theory described by an interaction

$$L_{\mathrm{I}} = \frac{-1}{4!}\,\lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{1.16}$$

must exhibit the property that

$$\bar{\lambda}^{(t)}_{ijkl}\phi_i\phi_j\phi_k\phi_l \geq 0 \tag{1.17}$$

for $t \to \infty$.)

Of course all arguments based upon low-order calculations of the beta function suffer from the same drawbacks as the corresponding calculation for QED. Namely, the region of interest [when $\lambda(t)$ becomes large] is clearly outside the domain of small λ where the lowest-order result for the beta function is valid.

A more stringent argument for triviality can be generated in an N-component $\mathrm{O}(N)$ symmetric ϕ^4 theory [1.64–1.66, 1.42, 1.67, 1.68] in the limit of large N. This theory is defined by an interaction Lagrange density

$$L_{\mathrm{I}}(\phi) = -\frac{\lambda}{4!}\,\frac{(\phi\cdot\phi)^2}{N}\,. \tag{1.18}$$

The beta function for the theory can be calculated by resumming an infinite series of diagrams to *all* orders in λ but only leading order in $1/N$. The result is [1.65, 1.66]

$$\frac{\mathrm{d}\bar{\lambda}(t)}{\mathrm{d}t} \equiv \beta_{\mathrm{O}(N)}(\lambda) = \frac{3\lambda^2(t)}{32\pi^2}\,, \tag{1.19}$$

which is algebraically the same as the lowest-order result for the $N = 1$ theory, eq. (1.13). What is new here is the inclusion of *all* orders in λ in the calculation of the beta function.

The easy criticism applied to eq. (1.14) [that as the ghost pole is approached, higher-order terms in λ become important] is not applicable to this large-N calculation. Moreover, for large N the location of the ghost pole implied by eq. (1.19) is independent of N. One might therefore conclude that λ_{R} must be zero for the theory to be consistent, i.e., that the $\mathrm{O}(N)$ ϕ^4 theory is trivial at least in the large-N limit.

This argument may be insufficient, however, because it is possible that the remainder term in the $1/N$ expansion is *nonuniform* in the limit of large t [1.70], and thereby invalidates the $1/N$ expansion. It is worth pointing out that the existence of a Landau ghost is *not* inconsistent with the development of a renormalized perturbation series, for a ghost pole appears in the exactly soluble Lee model [1.71–1.74, 1.57]. The point is that in the Lee model (which lacks antiparticles and crossing symmetry) the "bubble sum" is exact, and so the ghost appears in the full theory.

The addition of gauge fields to pure ϕ^4 theory is a necessity if results of practical interest to particle theorists are to be obtained. Moreover, the coupled gauge–Higgs system may be nontrivial even though the pure scalar theory *is* trivial. Examples of this phenomenon are discussed in refs. [1.75, 1.76] and section 5, but it is worthwhile mentioning that if the above $\mathrm{O}(N)$ scalar field is coupled to an $\mathrm{O}(N)$ gauge field, then in the large-N limit [1.62, 1.63] the theory is asymptotically free in both λ and the gauge coupling g (and thus presumably nontrivial) provided that

$$3\lambda_{\mathrm{R}}/g_{\mathrm{R}}^2 \leq \tfrac{3}{2}(1+\sqrt{2/3}) \sim 2.72\,. \tag{1.20}$$

The ratio eq. (1.20) is a familiar one, since to lowest order in the loop expansion the ratio of the mass m_{H} of the "Higgs" scalar to the mass m_{W} of the $N-1$ massive gauge bosons in the theory is given by

$$(m_{\mathrm{H}}/m_{\mathrm{W}})^2 = 3\lambda_{\mathrm{R}}/g_{\mathrm{R}}^2\,. \tag{1.21}$$

The plausible assumption (discussed further in section 5) that this asymptotically free fixed point is the only one in the theory then leads to an upper bound on its "Higgs" mass [1.76] for small g_{R},

$$(m_{\mathrm{H}}/m_{\mathrm{W}})^2 \leq \tfrac{3}{2}(1+\sqrt{2/3})\,. \tag{1.22}$$

An even more interesting scenario occurs [1.77, 1.80] in "eigenvalue" theories like the Georgi–Glashow model [1.79, 1.80] of the weak interaction (see also section 5). In these theories there exist fixed points where, if every coupling constant in the theory (including the "Yukawa" or fermion mass couplings) is chosen precisely, the theory is asymptotically free, and hence presumably nontrivial. As pointed out above, a theory in general need not be asymptotically free in order to be nontrivial, and the requirement that all its couplings be "fine tuned" is not generally considered an attractive possibility. Yet if only one ultraviolet fixed point existed in a theory, all particle masses and coupling constants would be determined by consistency conditions (reminiscent of the "bootstrap" hypothesis [1.81]), a provocative concept. This possibility also has a natural implementation in terms of the number of relevant directions at a fixed point in the Wilson renormalization group (see ref. [1.51] and section 3).

Obviously all such ideas are highly speculative, and further consideration of their validity demands that accurate nonperturbative methods for assessing the triviality or nontriviality of a theory be developed. It should, however, be clear that the problem of triviality is of more than academic interest, and a study of its implications may well lead to results of phenomenological import.

2. The precise approach: Rigorous formulation of triviality

It is well to keep in mind that before it can be determined whether or not a given field theory is trivial, it is first necessary to define (1) a field theory, and (2) triviality. The attendant terminological exactitude is not, however, a necessary precursor for comprehension of the remainder of this review. Readers uninterested in mathematical formalism thus may do well to skip this section and continue on to section 3. Those readers who, by contrast, are not disturbed by a precise approach to an already monumentally difficult problem are invited to continue reading this section. The first two subsections are brief interludes which sketch the definitions of quantum field theories and triviality. Following these is an introduction to rigorous results obtained largely by the powerful random-walk and random-current strategies. Finally a brief review is given of the construction of nontrivial three-, two-, and one-dimensional ϕ^4 field theories.

2.1. What is a quantum field theory?

The subject of axiomatic quantum field theory is largely beyond the scope of this review. Thus only a brief outline of a few major ideas in this subject appears below. Readers interested in further details may find refs. [2.1–2.4] useful.

A quantum field theory can be defined by a set of *axioms*—reasonable properties that a sensible theory should possess. One basic set is the Garding–Wightman definition of a Minkowski-space quantum field theory [2.5] (see also ref. [2.6]), which can be translated in terms of vacuum expectation values [2.7]. Field-theoretic vacuum expectation values W_n (also known as *Wightman distributions*) can be defined in terms of operators $\phi(x)$ acting on a vacuum state Ω,

$$W_n(x_1, \dots, x_n) = (\Omega, \phi(x_1) \cdots \phi(x_n)\Omega)\,. \tag{2.1}$$

This approach leads to the famous *Wightman Reconstruction Theorem* [2.2, 2.5], which states that, given a set of W_n obeying the Garding–Wightman axioms, there exists an essentially unique theory for which the W_n are the Wightman distributions.

The Wightman reconstruction theorem is based upon a Minkowski-space formulation of quantum field theory. However, much of the work pertinent to triviality is formulated in terms of Euclidean quantum field theory. Euclidean-space methods were first discussed in terms of perturbation theory [2.8–2.10]. Despite early work on Euclidean field theories [2.11, 2.12, 2.7] progress in this direction was largely initiated by Symanzik [2.13, 2.14] and by Nelson [2.15, 2.16], which inspired subsequent theorems by Osterwalder and Schrader [2.17]. (Although ref. [2.17] is correct in its general outlook, the reader should be warned that a technical error exists in the last lemma; see refs. [2.18] and [2.1].) The point of the Osterwalder–Schrader approach is that a unique Garding–Wightman theory can be reconstructed given an appropriate set of Euclidean region "Green's functions" (more commonly called *Schwinger functions*). These ideas were instrumental in developing a rigorous definition of quantum field theory based upon Euclidean-space path integrals.

Path integrals were first used "formally" by Feynman [2.20]. Subsequently [2.21] Kac developed rigorous aspects of the theory based upon considerations of the Ornstein–Uhlenbeck velocity process, which is in turn based upon the fundamental work of Wiener [2.22]. It remained for Nelson to show [2.15] that not only can one develop a path integral formalism over the free Minkowski field but that it is a manifestly Euclidean-invariant path integral (see also ref. [2.24]). Euclidean-space path integrals also allow a natural connection to be made with statistical mechanics and the renormalization group (see section 3).

2.2. *What does "triviality" mean?*

As is discussed below, much of the formal work done on triviality involves statements that a given theory is a trivial "generalized free field" if certain preconditions are met. The concept of a generalized free field was introduced in ref. [2.24], and can be expressed [2.1] in terms of a factorization property described below.

Consider the structure of $W_2(x, y) = \langle \Omega, \phi(x)\phi(y)\Omega \rangle$. By translation invariance, W_2 can be written as a function Δ_+ of $(x-y)$

$$W_2(x, y) = \Delta_+(x-y) . \tag{2.2}$$

From the Garding–Wightman axioms, it follows that $\Delta_+(x)$ has a Kallen–Lehmann representation [2.1, 2.25],

$$\Delta_+(x) = \int_{m^2>0} \Delta_+(x; m^2)\, \mathrm{d}\rho(m^2) , \tag{2.3}$$

where

$$\Delta_+(x; m^2) \equiv \int \frac{\mathrm{d}^3 p}{2E} \frac{\mathrm{e}^{\mathrm{i}p_\mu x^\mu}}{2\pi} , \quad E \equiv (\boldsymbol{p}^2 + m^2)^{1/2} , \tag{2.4}$$

and $\rho(m^2)$ is a polynomially bounded (positive) measure.

It is useful in the definition of a generalized free field to rely upon the notation [2.26]

$$[x_1, \ldots, x_n] = 0 \quad \text{if } n \text{ is odd} , \tag{2.5a}$$

$$[x_1, \ldots, x_n] \equiv \sum_{\text{pairs}} [x_{i_1}, x_{j_1}] \cdots [x_{i_n}, x_{j_n}] , \quad \text{if } n \text{ is even} , \tag{2.5b}$$

where $[x, y]$ denotes a given distribution in two variables, and the sum in eqs. (2.5b) is over all ways of writing $\{1, \ldots, 2n\}$ as $i_1 \ldots i_n \ldots j_1 \ldots j_n$ with $i_1 < i_2 < \cdots < i_n$, $i_1 < j_1, \ldots, i_n < j_n$. Then [2.1] it follows that if $[x, y] \equiv \Delta_+(y - x)$ possesses properties implied by the Garding–Wightman axioms, then a set of W_n defined by

$$W_n(x, \ldots, x_n) = [x_1, \ldots, x_n] \tag{2.6}$$

obeys these axioms and the associated field theory is a generalized free field, i.e., it is a trivial theory.

In the special case that $\rho(m^2) = \delta(m^2 - m_0^2)$, the associated theory is called a *free field of mass* m_0. The Schwinger functions for a free field are easy to find [2.1],

$$S_2(y; m_0^2) = \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\mathrm{e}^{\mathrm{i}k\cdot y}}{k^2 + m_0^2} = \frac{1}{2\pi} K_0(m_0|y|) \tag{2.7}$$

and

$$S_n(y_1, \ldots, y_n) = [y_1, \ldots, y_n]\,, \tag{2.8}$$

where K_0 is the associated Bessel function and

$$[x, y] = S_2(x - y; m_0^2)\,. \tag{2.9}$$

Discussions of free fields in the path integral formalism can be found in the standard reviews [2.19, 2.20, 2.1, 2.24].

2.3. Random walks and triviality

Many important questions about the triviality of ϕ^4 field theories are most naturally answered when the problem is formulated in terms of random walks. This *modus operandi* originated in the germinal work of Symanzik [2.14]; earlier pursuits of ϕ^4 triviality are reviewed in refs. [1.60] and [2.3].

Random-walk approaches and the related random-current ideas are an integral part of proofs of the triviality of ϕ^4 field theory in five and more dimensions [2.27–2.29] and in obtaining partial results in four dimensions [2.27, 2.28]. A heuristic summary of the latter partial results (reviewed below) is that one- and two-component pure ϕ^4 field theories in $d = 4$, constructed as a scaling limit of ferromagnetic lattice field theories, are trivial unless they are asymptotically free. Moreover [2.30], if these theories are to be nontrivial, mean field theory must be *exact* in these models. Obviously, if perturbation theory is any kind of a guide to these models, they are not asymptotically free (and logarithmic corrections to the mean field picture exist), but this has not been eliminated rigorously as a logical possibility. Further results of this random walk formulation can be found in refs. [2.30–2.37] (see also the reviews in refs. [2.38–2.42]).

The reformulation [2.13] of ϕ^4 field theories as random walks is a striking innovation. Some intuition as to how this comes about can be gleaned [2.27, 2.29, 2.40] from a simplified special case of the above calculations. Consider a scalar field theory defined upon a d-dimensional Euclidean hypercubic lattice Z_a^d with lattice spacing a. The fields ϕ_i obey a distribution law

$$\mathrm{d}\mu(\phi) = Z^{-1}\,\mathrm{e}^{-S(\phi)} \prod_i \mathrm{d}\phi_i\,, \tag{2.10a}$$

where Z is the standard partition function and

$$S(\phi) = \tfrac{1}{2}\beta \sum_{\langle ij\rangle} (\phi_i - \phi_j)^2 + \sum_i (\tfrac{1}{2} m_0^2 \phi^2 + \tfrac{1}{4}\lambda_0 \phi_i^4)\,. \tag{2.10b}$$

Here, $\langle ij\rangle$ refers to a sum over all nearest-neighbor pairs, each taken *once*. Equation (2.10b) can be rewritten

$$S(\phi) = \beta H(\phi) + \sum_i [\tfrac{1}{2}(m_0^2 + 2d\beta)\phi_i^2 + \tfrac{1}{4}\lambda_0\phi_i^4] , \quad H(\phi) \equiv + \sum_{\langle ij \rangle} \phi_i \phi_j , \tag{2.11a}$$

so that

$$\mathrm{d}\mu(\phi) = Z^{-1}\, \mathrm{e}^{-\beta H(\phi)} \prod_i \mathrm{d}\lambda(\phi_i) , \tag{2.11b}$$

$$\mathrm{d}\lambda(\phi_i) \equiv \exp[\tfrac{1}{2}(m_0^2 + 2d\beta)\phi_i^2 + \tfrac{1}{4}\lambda_0\phi_i^4] . \tag{2.11c}$$

The field strength renormalization is thus given by $\zeta(a) \equiv \beta(a)a^{2-d}$. As the continuum ($a \to 0$) limit is approached, the bare constants $\beta(a)$, $m_0(a)$, and $\lambda_0(a)$ are varied so as to obtain renormalized physical parameters λ_R and m_R which are finite.

In the free-field limit ($\lambda_0 \equiv 0$), the Schwinger function for the theory eq. (2.11) is given by

$$S_0(x, y) \equiv \langle \phi_x \phi_y \rangle_0 = \int \mathrm{d}\mu_0(\phi)\; \phi(x)\phi(y) = (-\beta\Delta + m_0^2)^{-1} , \tag{2.12}$$

which is defined upon Z_a^d in terms of a finite-difference Laplacian

$$-\beta\Delta + m_0^2 \equiv (2d\beta + m_0^2)I - J\beta . \tag{2.13}$$

Here I is the identity matrix and the elements J_{xy} of the matrix J are one if x and y are nearest neighbors and are zero otherwise. Then $S_0(x, y)$ can be expanded in a Neumann series in J,

$$S_0(x, y) = \beta^{-1} \sum_{n=0}^{\infty} \tilde{\beta}^{n+1} (J^n)_{xy} , \tag{2.14a}$$

$$\tilde{\beta} \equiv \beta(2d\beta + m_0^2)^{-1} . \tag{2.14b}$$

Consider next a random walk ω of length $|\omega|$. This walk can be thought of as being generated by a particle which makes steps of length one randomly in all $2d$ directions emanating from a given lattice site. Then $S_0(x, y)$ can be expressed in terms of a sum over all of these Brownian walks,

$$S_0(x, y) = \beta^{-1} \sum_{\omega : x \to y} \tilde{\beta}^{|\omega|+1} . \tag{2.15}$$

If $n_j(\omega)$ is the total number of visits of a given walk ω to some site j, then

$$S_0(x, y) = \beta^{-1} \sum_{\omega : x \to y} \prod_{J \in Z^d} \tilde{\beta}^{n_j(\omega)} . \tag{2.16}$$

Note that the critical value of $\tilde{\beta}$ (when the lattice spacing a vanishes) occurs when

$$(\tilde{\beta}_{\mathrm{crit}})^{-1} = 2d , \tag{2.17}$$

which is the number of directions that a particle at a given site can jump.

The more general case with λ_0 nonvanishing can also be treated by random-walk methods. This reformulation leads to a natural interpretation of triviality in terms of the intersection probability of random walks. It also leads to the above-mentioned theorems on ϕ^4 triviality in four and more dimensions. Define

$$\mathrm{d}\rho_n(s) \equiv \begin{cases} \delta(s)\,\mathrm{d}s & \text{if } n=0\,, \\ \dfrac{s^{n-1}}{(n-1)!}\,\theta(s)\,\mathrm{d}s & n=1,2,3,\ldots\,, \end{cases} \tag{2.18}$$

and

$$\mathrm{d}\rho_\omega(t) \equiv \prod_j \mathrm{d}\rho_{n_j(\omega)}(t_j)\,. \tag{2.19}$$

Note that t_j can be thought of as the total amount of "time" that the random walk ω spends at site j [where it makes $n_j(\omega)$ visits]. Define a t-dependent partition function

$$Z(t) \equiv \int \mathrm{e}^{-\beta H(\phi)} \prod_j g(\phi_j^2 + 2t_j)\,\mathrm{d}\phi_j\,, \tag{2.20a}$$

with [cf. eq. (2.11c)]

$$g(\phi^2) \equiv \mathrm{d}\lambda(\phi)/\mathrm{d}\phi\,. \tag{2.20b}$$

Then

$$S_\lambda(x,y) = \langle \phi_x \phi_y \rangle_\lambda = \sum_{\omega:\, x\to y} \beta^{|\omega|} \int z(t)\,\mathrm{d}\rho_\omega(t)\,, \tag{2.21a}$$

where

$$z(t) \equiv Z(t)/Z(0) > 0\,. \tag{2.21b}$$

In the noninteracting case $\lambda_0 = 0$,

$$z_0(t) = \prod_j \exp[-(2d\beta + m_0^2)t_j]\,, \tag{2.22}$$

so that [cf. eq. (2.12)]

$$S_0(x,y) = \langle \phi_x \phi_y \rangle_0 = (-\beta\Delta + m_0^2)^{-1}_{xy}\,. \tag{2.23}$$

The foregoing explication makes it clear that the random-walk model corresponding to the interacting ($\lambda_0 \neq 0$) theory has the same combinatoric structure as the free-field limit. What defines a particular class of walks are the weights $z(t)$ associated with each random walk when a sum over walks is taken.

It is easy to see that virtually any field-theoretic quantity can be expressed in terms of random walks. For instance, consider the connected four-point "Ursell" function defined by

$$u^{(4)}(x_1,x_2,x_3,x_4) \equiv \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle - \sum_P \langle \phi(P_1)\phi(P_2)\rangle\langle \phi(P_3)\phi(P_4)\rangle\,, \tag{2.24}$$

where the sum is over all pairings, P, of $\{1,2,3,4\}$. In the random-walk formalism this can be written [2.27, 2.29]

$$u^{(4)}(x_1,\ldots,x_4)=\sum_P \sum_{\omega_1,\omega_2}^{P} \beta^{|\omega_1|+|\omega_2|} \int \mathrm{d}\rho_{\omega_1}(t^1)\,\mathrm{d}\rho_{\omega_2}(t^2)\,[z(t^1+t^2)-z(t^1)z(t^2)]\,, \tag{2.25}$$

where $\Sigma^P_{\omega_1,\omega_2}$ ranges over all walks ω_1 and ω_2 with

$$\omega_1\colon x(P_1)\to x(P_2),\quad \omega_2\colon x(P_3)\to x(P_4)\,.$$

The renormalized quartic coupling λ_R can be defined in terms of the four-point function by

$$\lambda_\mathrm{R}\equiv -\bar{u}^{(4)}\chi^{-2}(m_\mathrm{R})^d\,, \tag{2.26}$$

where

$$\bar{u}^{(4)}\equiv\int \mathrm{d}^dx_2\,\mathrm{d}^dx_3\,\mathrm{d}^dx_4\,u^{(4)}(x_1,x_2,x_3,x_4)\,, \tag{2.27}$$

while the susceptibility χ is given by

$$\chi\equiv\int \mathrm{d}^dx\,S_\lambda(0,x)\,, \tag{2.28}$$

and

$$m_\mathrm{R}\equiv\lim_{|x|\to\infty}-\frac{1}{|x|}\ln S_\lambda(0,x) \tag{2.29}$$

defines the renormalized mass m_R.

An upper bound on the four-point function (essential for the following discussion on triviality) can now be constructed. Define

$$\mathring{z}(t)\equiv\prod_i \mathrm{e}^{\lambda_0 t_i^2}z(t)\,. \tag{2.30}$$

If $\ln\mathring{z}(t^1+t^2)$ is written as an integral over derivatives in t^1 and t^2 and Ginibre's inequality [2.43] is used to estimate the integrand, it can be shown that [2.27]

$$\mathring{z}(t_1+t_2)\geq\mathring{z}(t_1)\mathring{z}(t_2)\,. \tag{2.31}$$

Thus it follows that (recall that $u^{(4)}$ is negative)

$$u^{(4)}(x_1,\ldots,x_4)\geq\sum_P\sum_{\omega_1,\omega_2}\beta^{|\omega_1|+|\omega_2|}\int \mathrm{d}\rho_{\omega_1}(t^1)\,z(t^1)\,\mathrm{d}\rho_{\omega_2}(t^2)z(t^2)\left[\exp\!\left(-2\lambda_0\sum_i t_i^1t_i^2\right)-1\right]. \tag{2.32}$$

Note that if j is not contained within the random walk ω_i, then t^i_j vanishes. Thus the right-hand side vanishes unless a random walk ω_1 intersects a random walk ω_2 with nonvanishing probability. This is a major result of the formalism. The renormalized quartic coupling can be bounded from above:

$$\lambda_R \le 3P_{\text{int}}\,, \tag{2.33}$$

where P_{int} is the *average intersection probability* for two "field-theoretic" walks ω_1 and ω_2 with weights $z(\omega_1)$ and $z(\omega_2)$ as defined above. Moreover, it can be shown that (see, e.g., ref. [2.40])

$$0 \le \langle \phi(x)\phi(y)\rangle < c\beta^{-1}|x-y|^{2-d}\,, \tag{2.34}$$

which says, heuristically, that the Hausdorff dimension of these field-theoretic walks is at most two. Thus it may be intuited that P_{int} vanishes in more than four dimensions. The motivation for this noumenon is the fact that walks of Hausdorff dimension two fill a plane (heuristically speaking). Thus since two planes in general do not intersect in more than four dimensions, neither will two such random walks. The ordinary random walk (corresponding to a free field theory) has Hausdorff dimension two, for instance, and is known to be nonintersecting in $d>4$ [2.44, 2.30, 2.31].

Another pictorial example of some utility is the limit obtained by analytic continuation of an N-component ϕ^4 field theory to $N=0$. It is well known (see, e.g., ref. [2.45]) that this limit of pure ϕ^4 field theory is the Edwards model [2.46] of the self-suppressing random walk, which becomes the self-avoiding walk (SAW) in the nonlinear sigma model limit of large λ_0. Numerical simulations of this walk [2.30, 2.47] can be performed with remarkable precision. The critical functions of this theory can thus be measured accurately; the results are consistent with the idea that this limit model is trivial in four and more dimensions. In particular, these simulations predict logarithmic violations of mean-field scaling for the renormalized mass. The significance of scaling violations is further elaborated upon below.

It is therefore not surprising that the above inequality eq. (2.33) can be used [2.27, 2.28] to show that the connected four-point function vanishes in *five and more dimensions* for one- and two-component ϕ^4 field theories. In fact, inequalities of this form can be generated and used to show that all connected $2n$-point Wightman distributions vanish in $d>4$ (for $n=2,3,\ldots$). Thus these theories are generalized free fields (and presumably are also free fields).

It also follows [2.27, 2.28] that in four dimensions one- and two-component ϕ^4 field theories are trivial, i.e., that at noncoinciding arguments

$$\lim_{a\to 0} u^{(4)}(x_1, x_2, x_3, x_4) = 0\,, \tag{2.35}$$

provided that in this limit of vanishing lattice spacing a the field strength renormalization vanishes:

$$\lim_{a\to 0} \zeta(a) = 0\,. \tag{2.36}$$

If eq. (2.36) is *not* satisfied (i.e., if $\dim[\phi]=1$) the limiting theory cannot be both Euclidean and scale invariant unless it is a free field theory [2.48]. Thus if ϕ^4 field theory (with $N=1$ or 2) is actually nontrivial in four dimensions, it must then be *asymptotically free*, in sharp contrast to the expectations generated by a perturbative evaluation of the beta function. In other words, the beta function has no nontrivial zeros (see section 1).

More stringent results have also been obtained within this framework. It has been shown that [2.30, 2.31]

$$0 \le \lambda_R \le \text{const. } \beta^2 \cdot (m_R)^d \, \partial\chi/\partial\beta \, . \tag{2.37}$$

In four dimensions, when η ($=\gamma_\phi$, the anomalous dimension of ϕ) vanishes, eq. (2.37) implies that λ_R vanishes as the critical surface is approached unless mean field theory is exact, i.e., unless, as $\beta \to \beta_c$ from below,

$$m_R \sim (\beta_c - \beta)^\nu \, , \qquad \chi \sim (\beta_c - \beta)^\gamma \, , \tag{2.38}$$

with $\nu = \frac{1}{2}$ and $\gamma = 1$. Corrections to eq. (3.38) can be at most logarithmic [2.30, 2.35]; such logarithmic corrections to scaling are in fact predicted by the renormalization group (see section 3 and refs. [2.49–2.51]), and are also suggested by the above-mentioned numerical data for the self-avoiding random walk. Moreover, if these logarithmic corrections are present, then there is no broken-symmetry phase of the theory [2.40, 2.31].

2.4. *Nontriviality of ϕ^4 field theory when $d < 4$*

For the sake of completeness, a few references on the construction of a nontrivial ϕ^4 field theory in three and two dimensions are given. The first and most important steps in the construction of the three-dimensional model were taken by Glimm and Jaffe [2.52]. Later work in this direction includes refs. [2.53]. Other approaches may be loosely categorized as the "Italian" method [2.54], Balaban's method [2.55], the Battle–Federbush method [2.56], and the Brydges–Frohlich–Sokal method [2.31]. This last method is particularly simple, and does not make use of the "phase cell analysis", which the other schemes are based upon (see also section 3). Several clever (if incomplete) arguments are also given in ref. [1.60].

It should be pointed out that the construction of ϕ^4 field theory in *three* dimensions is a problem of extraordinary difficulty, since infinite mass and vacuum energy divergences appear. In *two* dimensions, by contrast, these divergences can be eliminated by the simple expedient of Wick ordering. Traditional constructions of two-dimensional ϕ^4 field theory can be found in refs. [2.1, 2.3, 2.29], see also refs. [2.31] and [1.60].

One-dimensional ϕ^4 field theory has also been constructed, see ref. [2.57].

3. The modern approach: Triviality and the renormalization group

3.1. *Motivation*

The casual reader may well believe by this point that the study of triviality in quantum field theories is an esoteric exercise in formal mathematics. In this section the fundamental notions of triviality are recast in terms of the tenets of the renormalization group, with the hope of rescuing triviality from this innocuous desuetude. Many of the concepts of triviality are in fact most naturally expressed within this framework. Moreover, the renormalization group study of triviality can be accomplished with some efficacy by numerical analyses of lattice field theories.

The renormalization group is more of a way of thinking about a problem than a specific technique. Indeed, the sobriquet "renormalization group" is presently applied to a number of ideas and calculational methods which can appear entirely disjoint to the uninitiated. The major theme of all such approaches is the idea that in a system with a large or (as in the case of a quantum field theory) an infinite number of degrees of freedom, the physical properties of this system can be equated to those of a system with fewer degrees of freedom but with different interactions. This idea suggests that the original theory can be successively mapped onto a sequence of theories, each with fewer degrees of freedom than its predecessor, and each characterized by a new set of coupling constants. The implied sequence of sets of coupling constants defines a renormalization group "flow" or trajectory as the number (or number density) of degrees of freedom is steadily reduced. Instead of solving the system directly, its properties are revealed from an analysis of these flows. In the case of a quantum field theory, useful quantities such as the number of independent renormalized parameters in the theory or the anomalous dimensions of various operators can be deduced in this fashion. Moreover, something calculated in one region in the space of coupling constants (e.g., where perturbation theory is valid) can be transported to another region of coupling constants by integrating the result along a renormalization group trajectory. (Such is the origin of "renormalization group improved perturbation theory").

As discussed in section 1, the original (or "historical") renormalization group [1.44–1.50] had its origins in the fact that in a "sensible" quantum field theory, measurable quantities can be defined (in terms of other measurable quantities) in a fashion which is independent of the renormalization point μ. This renormalization group formalism is quite familiar to high-energy physicists, and is thus discussed below only in reference to more contemporary approaches. Readers interested in further details may enjoy perusing refs. [1.49], [1.50], and [3.1].

The more modern renormalization group due primarily to Wilson [1.51, 1.52] draws heavily upon these ideas. This renormalization group has largely been applied in statistical mechanics, though recently it has come to be useful in quantum field theories defined upon a spacetime lattice (see ref. [3.23] and section 5). In this approach, no true "renormalized" couplings are ever calculated. Instead, the Wilson renormalization program is applied directly to the bare cutoff field theory. The high-momentum components of the path integral are integrated out, and the cutoff Λ is reduced. The action for the theory changes its functional form under this operation, and in general an infinite set of couplings is generated. This is in distinct contrast to the historical renormalization group, where the number of couplings does not change as the renormalization point μ is varied. Nevertheless there are points of contact between the two approaches. One example is the fact that the number of directions of instability at a fixed point in the Wilson approach equals the number of independent renormalized couplings for the theory used in the historical approach. The application of these ideas to ϕ^4 field theory is discussed below. As the Wilson approach is the most useful in the study of triviality, it is discussed first, and then its relations to other approaches are explored.

3.2. The Wilson renormalization group

3.2.1. Conceptual foundations

It is easiest to begin with the formulation of a single-component ϕ^4 theory on a spacetime lattice. Consider first the theory in continuous d-dimensional Euclidean space, where it can be defined by the partition function (or vacuum-to-vacuum transition amplitude)

$$Z = \int \mathrm{D}\phi \, \exp(-S) = {}_{t=\infty}\langle 0 \,|\, 0\rangle_{t=-\infty} \,, \tag{3.1a}$$

where

$$S = \int \mathrm{d}^d x [\tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2} m_0^2 \phi^2 + \lambda\phi^4] \,, \tag{3.1b}$$

with $\phi = \phi(x)$. One lattice formulation of the model is given by the association of a canonically dimensionless field S_n with points n in a hypercubic lattice. The discrete approximations

$$\phi(x) \to a^{1-d/2} S_n\,, \qquad \partial_\mu \phi(x) \to a^{-d/2}(S_{n+\mu} - S_n)\,, \qquad \int \mathrm{d}^d x \to a^d \sum_n\,, \tag{3.2a}$$

along with the redefinition of the theory in terms of dimensionless couplings u and r,

$$\lambda \equiv a^{2(d-4)} u\,, \qquad m_0^2 \equiv a^{-2} r \tag{3.2b}$$

(where a is the lattice spacing and $\hat{\mu}$ is a unit vector in the direction μ) can be applied to the action eq. (3.1b) to yield

$$\begin{aligned} S_{\text{lattice}} &= \sum_{n,\mu} \tfrac{1}{2}(S_{n+\mu} - S_n)^2 + \sum_n (\tfrac{1}{2} r S_n^2 + u S_n^4) \\ &= -\sum_{n,\mu} S_n S_{n+\mu} + u \sum_n (S_n^2 - K)^2 + \text{constant}\,, \end{aligned} \tag{3.3}$$

where $K = -r/4u$. In the limit of large λ, the theory defined by eq. (3.3) is (modulo a rescaling $S_n \to \sqrt{K}\,\sigma_n$) an Ising model, defined by the partition function

$$\begin{aligned} Z_{\text{Ising}} &= \prod_n \left(\int \mathrm{d}\sigma_n\, \delta(\sigma_s^2 - 1) \right) \exp\left(K \sum_{n,\,\mu} \sigma_n \sigma_{n+\mu} \right) \\ &\equiv \int \mathrm{D}\sigma \exp(-S_{\text{Ising}})\,, \end{aligned} \tag{3.4}$$

whose correlation functions are defined by

$$\Gamma_n = \langle \sigma_0 \sigma_n \rangle \equiv Z^{-1} \int \mathrm{D}\sigma\, (\sigma_0 \sigma_n) \exp(-S_{\text{Ising}})\,. \tag{3.5}$$

Note that the (σ) are restricted to the values ± 1. The asymptotic behavior of Γ_n for large $|n|$ is thought to be

$$\Gamma_n \sim \exp[-|n|/\xi(K)]\,, \tag{3.6}$$

where $\xi(K)$ is a dimensionless "correlation length". It is convenient to define a renormalized mass for the scalar particles of the theory in terms of $\xi(K)$,

$$m_{\mathrm{R}} = [a\xi(K)]^{-1}\,, \tag{3.7}$$

whereupon it becomes evident that if a continuum limit $(a \to 0)$ of the theory with particles of finite mass exists for some value $K = K_{\mathrm{c}}$, then $\xi(K_{\mathrm{c}})$ must be infinite. This requirement of a divergent dimensionless correlation length implies that a continuum limit of a lattice theory must occur at a phase transition of second (or higher) order. The analysis of a continuum limit of a lattice theory is thus a study of critical phenomena, for which Wilson's renormalization group was designed.

A brilliant qualitative picture of the behavior of the Ising model at a critical point was suggested by Kadanoff [3.3]. This idea describes the behavior of a system with large correlation length in terms of the behavior of a system with small correlation length. (The following discussion is actually [1.51] a rephrasing of Kadanoff's ideas.) Consider for simplicity a plane lattice of Ising spins (fig. 3.1) divided into blocks of four spins each. Near a critical point, ξ is very large, so the four Ising spins in a block are very well correlated. In particular, the four spins in a block act much like a single spin—all four spins will be either up or down. The original lattice can thus be replaced by an effective lattice where the interactions are between blocks of spins rather than between the spins themselves. The correlation length for the block system is half the correlation length of the original (or "site") system. This process of blocking can be repeated indefinitely until a system with $\xi \sim 1$ is reached.

Suppose that in the original system K is a certain value K_0, and the blocked system can be described with a nearest-neighbor action with $K = K_1$. Such a truncation of the blocked action is at best a severe and dubious approximation. However, it is easy (at least in principle) to keep track of the arbitrarily large number of other parameters, and so in eqs. (3.8) they are neglected. The correlation length in the original and blocked system are then related by

$$\xi(K_1) = \tfrac{1}{2}\xi(K_0)\,, \tag{3.8a}$$

where the "truncation" assumptions

$$K_1 = f(K_0) \tag{3.8b}$$

and

$$K_n = f(K_{n-1}) \tag{3.8c}$$

are made, so that (blocking n times)

$$\xi(K_n) = 2^{-n}\xi(K_0)\,. \tag{3.8d}$$

The fact that at $K = K_c$ there is a second-order phase transition with $\xi(K_c)$ infinite means that

$$f(K_c) = K_c\,. \tag{3.9}$$

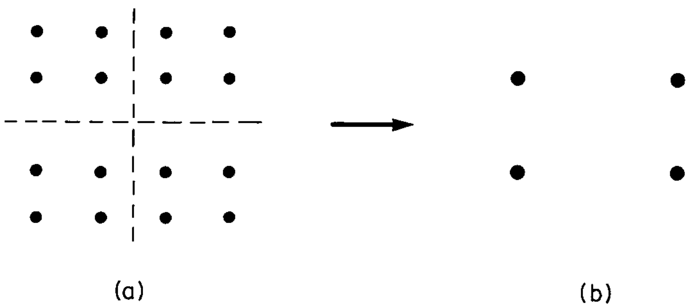

Fig. 3.1. Schematic diagram displaying the concept of interactions between spins being equivalent to interactions among blocks of spins at a critical point. (a) The original lattice, (b) the block lattice.

In other words, K_c is a *fixed point* of the transformation eq. (3.8c). The study of such fixed points is at the heart of the renormalization group program. When K is near K_c, $f(K)$ can be expanded in a Taylor series,

$$f(K) - K_c = \lambda(K - K_c)\,, \tag{3.10}$$

where $\lambda = \mathrm{d}f(K)/\mathrm{d}K$ at K_c. Since

$$\xi(K) \propto (K - K_c)^{-\nu} \tag{3.11}$$

for K near K_c, then [1.51] in this limit

$$2 = \frac{\xi[f(K)]}{\xi(K)} = \left(\frac{f(K) - K_c}{K - K_c}\right)^{\nu}. \tag{3.12}$$

From eq. (3.10) it follows that

$$\nu = \ln 2/\ln \lambda\,. \tag{3.13}$$

This critical exponent ν is related to the anomalous dimension γ_{ϕ^2} of the operator ϕ^2 in the original ϕ^4 theory, as is discussed below. Thus from the recursion relation $f(K)$ it is possible to determine the critical point K_c of the theory as well as the anomalous dimensions of various operators in the associated field theory. The important issue to address is therefore how the function $f(K)$ can be calculated, i.e., how the block action governing the block system is determined from the site system. Note that this function can produce nonanalytic behavior in $\xi(K)$ even if $f(K)$ is analytic near K_c.

The above heuristic scheme is now explored in detail through several examples. These examples allow an explication of the Wilson approach to renormalization, which allows one to calculate the renormalization group trajectories [defined above for a single coupling K, by the function $f(K)$]. Although the existence of such trajectories is implicit within Kadanoff's ideas, it remained for Wilson to develop this calculational scheme.

3.2.2. *Application to ϕ^4 field theory: The gaussian fixed point*

Consider the n-component ϕ^4 theory defined by the action

$$S = \int \mathrm{d}^d x\, [\tfrac{1}{2}m_0^2(\phi\cdot\phi) + \lambda_0(\phi\cdot\phi)^2 + \tfrac{1}{2}c(\partial_\mu\phi\cdot\partial^\mu\phi)] \tag{3.14a}$$

$$= \tfrac{1}{2}\sum_{i,k}(m_0^2 + ck^2)|\tilde{\phi}_{ik}|^2 + \lambda_0 L^{-d}\sum_{k,k',k'',ij}\tilde{\phi}_{i,k}\tilde{\phi}_{i,k'}\tilde{\phi}_{j,k''}\tilde{\phi}_{j,-(k+k'+k'')}\,, \tag{3.14b}$$

where i and j are the indices of the components of $\tilde{\phi}_k$, the Fourier transformation of $\phi(x)$. All momenta are restricted to magnitudes less than a cutoff Λ (i.e., $|k| < \Lambda$). Here, L is the length of a side of the "box" in which the theory is defined (the infrared cutoff).

The renormalization group analysis performed below (following ref. [3.6]) on eqs. (3.14) is defined for an arbitrary rescaling factor $b > 1$. Both the scalar fields ϕ and the spacetime in which the theory is embedded are continuous. The blocking procedure can be summarized in two steps.

Step 1 (Kadanoff transformation): The components $\tilde{\phi}_q$ for $\Lambda/b < |q| < \Lambda$ are integrated out. This is essentially the same procedure as dividing an Ising system into blocks and averaging the site spins over a block.

Step 2 (Wilson definition of block fields): The block fields ϕ'_k are defined in terms of the remaining original fields $\{\phi_k : |k| < \Lambda/b\}$ by

$$\phi'_{k'} = (\rho_b b^{d/2})\phi_{bk} , \tag{3.15a}$$

so that $0 < |k'| < \Lambda$. In order that the transformation can be repeated *ad infinitum*, it is necessary that ρ_b be a power of b. It is traditional to define an exponent η via

$$\rho_b b^{d/2} \equiv b^{1-\eta/2} . \tag{3.15b}$$

As is shown below, the exponent η is equal to the anomalous dimension γ_ϕ of the field ϕ.

Two alternate explanations of the rescaling factor $b^{1-\eta/2}$ can be advanced, depending upon the background of the reader. The more glib explanation can be offered to the particle theorist: As the cutoff is reduced, the field strength renormalization changes. (In fact it is in general impossible to obtain a fixed point of the renormalization group transformation *without* such rescaling.)

The explanation for the condensed matter theorist is more intuitive. Recall the heuristic Ising model study of renormalization given above. If the only allowed configurations are those in which all site spins in a block are aligned, then the sum of the four spins in a block takes on the values ± 4. Rescaling by a factor of four is needed to ensure that the blocked system is also an Ising model, so that the transformation can be repeated and fixed points obtained. (Actually configurations in which all site spins in a block are *not* aligned are important, so a nonlinear blocking transformation is needed [3.5].)

The two steps defining the block renormalized action S' can be combined to give

$$\exp(-S' - AL^d) = \left(\int \mathrm{D}\bar{\phi}\, \mathrm{e}^{-S}\right)_{\tilde{\phi}_k \to b^{1-\eta/2}\tilde{\phi}_{bk}} , \tag{3.16a}$$

where

$$\int \mathrm{D}\bar{\phi} \equiv \int \prod_{\Lambda/b<|q|<\Lambda} \prod_i \mathrm{d}\tilde{\phi}_{iq} , \tag{3.16b}$$

and S' has been defined so as to include no additive constant.

It is simpler to consider first the case with λ equal to zero. The required integrals are simple gaussians, and give the result

$$\begin{aligned} S' &= \frac{1}{2} \sum_{i,|k|<\Lambda/b} b^{2-\eta}(ck^2 + m^2)|\tilde{\phi}_{i,bk}|^2 \\ &= \frac{1}{2} \sum_{i,\,|k'|<\Lambda} b^{-\eta}(ck'^2 + m^2b^2)|\tilde{\phi}_{i,k'}|^2 , \end{aligned} \tag{3.17a}$$

$$AL^d = -\frac{1}{2} \sum_{\Lambda/b<|q|<\Lambda} n \ln\left(\frac{2\pi}{m^2 + cq^2}\right) . \tag{3.17b}$$

Equation (3.17a) is the same as eq. (3.14b), but with new parameters

$$m^2 \cdot b^{2-\eta} \equiv (m')^2 , \tag{3.18a}$$

$$cb^{-\eta} \equiv c' . \tag{3.18b}$$

The renormalization group transformation R_b can be written symbolically

$$R_b \mu = \mu' \tag{3.19a}$$

as an abbreviation for

$$\begin{pmatrix} b^{2-\eta} & 0 \\ 0 & b^{-\eta} \end{pmatrix} \begin{pmatrix} m^2 \\ c \end{pmatrix} = \begin{pmatrix} m'^2 \\ c' \end{pmatrix} . \tag{3.19b}$$

Fixed points μ^* of R_b are defined by

$$R_b \mu^* = \mu^* . \tag{3.20}$$

The so-called "gaussian" fixed point of interest to field theorists occurs in eq. (3.19b) when $\eta = 0$,

$$\mu_0^* = \begin{pmatrix} 0 \\ c \end{pmatrix} . \tag{3.21}$$

By referring back to eq. (3.14), it can be seen that this fixed point corresponds to a free field theory.

When $\eta = 2$ another fixed point appears in the theory if $m^2 > 0$,

$$m^2 > 0: \quad \mu_\infty^* = \begin{pmatrix} m^2 \\ 0 \end{pmatrix} . \tag{3.22a}$$

If m^2 is negative, an (infinitesimal) quartic coupling constant can be used to stabilize the theory, and a third fixed point appears,

$$m^2 < 0: \quad \mu_{-\infty}^* = \begin{pmatrix} m^2 \\ 0 \end{pmatrix} . \tag{3.22b}$$

The three fixed points μ_0^*, μ_∞^*, and $\mu_{-\infty}^*$ have very different interpretations. The fixed point μ_0^* is the critical fixed point of the theory (corresponding to the ferromagnetic phase transition of the associated Ising model) and is the only fixed point of field-theoretic significance. The fixed point μ_∞^* describes a situation in which each block field has a gaussian distribution and is independent of all other blocks. It is analogous to the high-temperature limit of the Ising model. The fixed point $\mu_{-\infty}^*$ (corresponding to the low-temperature limit of the Ising model) describes a system in which each block field takes the same large value. Neither μ_∞^* nor $\mu_{-\infty}^*$ describes a system with an infinite correlation length. Thus, by the discussion following eq. (3.7), they are unimportant for field theory and are not considered further.

How does the correlation length ξ scale under the renormalization group transformation? Define a correlation function $G_c^{(2)}(x)$ by

$$G_c^{(2)}(x) \equiv \langle \phi(x)\phi(0) \rangle - \langle \phi(x) \rangle \langle \phi(0) \rangle , \tag{3.23a}$$

with

$$\langle O(x)\rangle \equiv \frac{\int \mathrm{D}\phi\, \mathrm{e}^{-S} O(x)}{\int \mathrm{D}\phi\, \mathrm{e}^{-S}}\,. \tag{3.23b}$$

An oft-used definition of the dimensionless correlation length ξ is then

$$\xi^2 = \Lambda^2\, \frac{\int x^2 G_\mathrm{c}^{(2)}(x)\,\mathrm{d}^d x}{\int G_\mathrm{c}^{(2)}(x)\,\mathrm{d}^d x}\,, \tag{3.24}$$

i.e., ξ is an "effective range of correlation". For S given by eqs. (3.14) with λ set to zero,

$$\xi^2 \propto (m^2 - m^{*2})^{-\nu} = (m^2)^{-1/2} \tag{3.25a}$$

(since m^* equals zero in this case) and at the gaussian fixed point μ_0^*,

$$\xi'(R_b\mu) \propto (m'^2 - m^{*2})^{-\nu} = (m'^2)^{-1/2} = \xi(\mu)/b\,. \tag{3.25b}$$

By following the argument leading to eq. (3.13), it can be seen that

$$\nu = \frac{\ln b}{\ln(b^{2-\eta})} = \frac{1}{2}\,. \tag{3.26}$$

From eqs. (3.15) and (3.23) it follows that the Fourier transform $\tilde{G}_\mathrm{c}^{(2)}(k)$ of $G_\mathrm{c}^{(2)}(x)$ varies under the renormalization group transformation like

$$\tilde{G}_\mathrm{c}^{(2)}(k, \mu) = b^{2-\eta}\tilde{G}_\mathrm{c}^{(2)}(bk, R_b\mu)\,, \tag{3.27}$$

where the dependence upon the parameters μ has been made explicit. Since eq. (3.27) holds for any b, it is permissible to choose $b = k^{-1}$, so that at a fixed point (where ξ is infinite),

$$\tilde{G}_\mathrm{c}^{(2)}(k, \mu^*) = k^{-2+\eta}\tilde{G}_\mathrm{c}^{(2)}(1, \mu^*)\,, \tag{3.28}$$

so that the anomalous dimension γ_ϕ of the field ϕ is given by

$$\gamma_\phi = \eta\,. \tag{3.29a}$$

Moreover, as is shown below [see eq. (3.12)], the anomalous dimension of the operator ϕ^2 is given by

$$\gamma_{\phi^2} = 2 - \nu^{-1}\,, \tag{3.29b}$$

and thus vanishes at μ_0^*. Since the anomalous dimension γ_ϕ is also zero at the gaussian fixed point, the renormalization group analysis is consistent with the statement that this point describes a free field theory. In this particular case, the result is obvious by inspection of eqs. (3.14). Nevertheless it is useful to demonstrate the formalism in preparation for study of more complicated systems.

In the most general case, there are an infinite number of interactions, and μ is a vector with an infinite number of components. The *critical surface* of a fixed point μ^* (where the correlation length is infinite, and hence a renormalized quantum field theory can exist) is defined as the set of points μ_c such that

$$\lim_{b\to\infty} R_b\mu_c = \mu^* . \tag{3.30}$$

If μ is near μ^*, it makes sense to expand

$$\mu = \mu^* + \delta\mu , \tag{3.31}$$

where $\delta\mu$ is "small". Then

$$\delta\mu' \simeq R_b^L\, \delta\mu , \tag{3.32}$$

where (assuming that the renormalization group transformation is nonsingular)

$$(R_b^L)_{\alpha\beta} \equiv (\partial\mu'_\alpha/\partial\mu_\beta)_{\mu^*} \tag{3.33}$$

is a linear operator. If R_b^L has eigenvectors v_j corresponding to eigenvalues $e_j(b)$, then, since

$$R_b^L R_{b'}^L v_j = R_{bb'}^L v_j , \tag{3.34}$$

it follows that

$$e_j(b)e_j(b') = e_j(bb') . \tag{3.35}$$

and hence

$$e_j(b) = b^{y_j} , \tag{3.36}$$

where y_j is independent of b.

The vectors $\delta\mu$ and $\delta\mu'$ can be expanded in terms of the v_j,

$$\delta\mu = \sum_j t_j v_j , \tag{3.37a}$$

$$\delta\mu' = \sum_j t'_j v_j , \tag{3.37b}$$

$$t'_j = b^{y_j} t_j . \tag{3.37c}$$

Each of the t_j is *by assumption* a smooth function of the couplings in the theory. It is also clear that for each of the y_j that are positive (and thus define what are commonly known as "relevant" directions at the fixed point) the corresponding t_j must vanish on the critical surface. The t_j corresponding to *negative* eigenvalues y_j define "irrelevant" interactions, and in general do not vanish on the critical surface. Finally, the y_j which vanish define "marginal" directions. These marginal eigenvalues are something of a nuisance, and often mean that a more careful analysis of the problem (i.e., the inclusion of more terms in a power series) is needed. They can also lead to the presence of logarithmic violations of scaling relations like eq. (3.28) (see section 5.2).

The above concepts can be illustrated by a perturbative analysis in λ near the fixed point μ_0^*. The result for small λ is [3.4, 1.51, 3.6–3.8]

$$\begin{aligned} \lambda' &= b^{4-d}\lambda\,, \\ (m')^2 &= b^2[m^2 + \lambda(\tfrac{1}{2}n+1)(n_c/n)(1-b^{2-d})]\,, \\ c' &= c\,, \end{aligned} \tag{3.38}$$

where

$$n_c = \frac{n}{c} K_d \frac{\Lambda^{d-2}}{d-2}\,, \tag{3.39a}$$

$$K_d = \frac{2^{1-d}\pi^{-d/2}}{\Gamma(\frac{1}{2}d)} \tag{3.39b}$$

$$= \frac{1}{8\pi^2} \quad \text{if } d=4\,. \tag{3.39c}$$

Here, n is the number of components of the ϕ^4 theory and K_d is the surface area of a unit sphere in d-dimensional space divided by $(2\pi)^2$.

The matrix R_b^L is then given by

$$R_b^L = \begin{pmatrix} b^2 & (b^2 - b^{4-d})B & 0 \\ 0 & b^{4-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.40a}$$

where (near μ_0^*)

$$\begin{pmatrix} m'^2 \\ \lambda' \\ c' \end{pmatrix} = R_b^L \begin{pmatrix} m^2 \\ \lambda \\ c \end{pmatrix}, \tag{3.40b}$$

and

$$B \equiv (\tfrac{1}{2}n+1)n_c/n\,. \tag{3.40c}$$

The eigenvalues of R_b^L are b^2, b^{4-d}, and 1. The eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -B \\ 1 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{3.41}$$

and correspond to exponents

$$y_1 = 1/\nu = 2\,, \qquad y_2 = 4-d\,, \qquad y_3 = 0 \tag{3.42}$$

In the neighborhood of the fixed point μ_0^*, the vector μ of coupling constants is given by [see eqs. (3.37)]

$$\mu = \mu_0^* + t_1 v_1 + t_2 v_2 + t_3 v_3 , \tag{3.43a}$$

with

$$\mu_0^* = \begin{pmatrix} (m^2)^* \\ \lambda^* \\ c^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} , \tag{3.43b}$$

$$t_1 = m^2 + B\lambda , \qquad t_2 + \lambda , \qquad t_3 = c \tag{3.43c}$$

The equation for the critical surface μ_c is simply $t_1 = 0$. If the spacetime dimensionality d of the system is greater than four, the parameter λ is irrelevant along this critical surface. (Note that, since the anomalous dimensions γ_ϕ and γ_{ϕ^2} vanish for any infinite-cutoff theory defined on this surface, such a theory must be trivial.) These renormalization group flows point away from this critical surface when d is less than four, implying that another fixed point must exist. Both results are consistent with the theorems discussed in section 2, which show that ϕ^4 field theory is trivial (nontrivial) when d is greater than (less than) four. (Note that this simple analysis is inconclusive when d equals four.)

A good question to ask is whether other terms can be present in the original action S [of eq. (3.14)] which generate further relevant interactions at the gaussian fixed point. It is easy to see, however, that if another interaction is included with P_ϕ powers of ϕ and P_∂ powers of ∂_μ, then the corresponding eigenvalue is $y = P_\phi(1 - d/2) - P_\partial + d$. In four dimensions the symmetries of Lorentz invariance and reflection ($\phi \to -\phi$) symmetry in ϕ forbid the existence of terms with positive y in the action. Thus within the insular domain of validity of perturbation theory, the above analysis of the gaussian fixed point is complete.

3.2.3. The renormalization group in the large-n limit

Another tractable example which demonstrates the machinery of the renormalization group is the limit of a ϕ^4 theory as the number n of components increases without bound. The action for this large-n limit can be written

$$S = \int \mathrm{d}^d x \, [\tfrac{1}{2} c(\partial\phi)^2 + U(\phi^2)] , \tag{3.44}$$

where $U(\phi^2)$ is an arbitrary function. The analysis presented here follows refs. [3.9]. Since for large n the fractional fluctuation is small

$$\frac{\phi^2 - \langle \phi^2 \rangle}{\langle \phi^2 \rangle} \ll 1 , \tag{3.45}$$

it is possible to expand

$$U(\phi^2) \approx U(\langle \phi^2 \rangle) + \tfrac{1}{2} t(\langle \phi^2 \rangle)(\phi^2 - \langle \phi^2 \rangle) , \tag{3.46}$$

where

$$t(\phi^2) = 2\partial U(\phi^2)/\partial\phi^2 , \tag{3.47}$$

so that for large n

$$S \approx \tfrac{1}{2} \int \mathrm{d}^d x \, [c(\partial\phi)^2 + t(\langle\phi^2\rangle)\phi^2] + L^d[U(\langle\phi^2\rangle) - \tfrac{1}{2}t(\langle\phi^2\rangle)\langle\phi^2\rangle] \,. \tag{3.48}$$

The first term in eq. (3.8) is quadratic in ϕ and the second is a constant. The probability distribution $\exp(-S)$ is thus a gaussian, and the renormalization group transformation can be performed analytically. (Note that the function $t(\langle\phi^2\rangle)$ plays the role of an effective m^2, much like a self-consistent field approximation.) After the blocking (3.16) is performed on the action (3.44), the block action is of the same form, but with $c' = cb^{-\eta}$. Therefore a fixed point with finite c requires that

$$c' = c \,, \qquad \eta = 0 \,. \tag{3.49}$$

A simple way of presenting the solutions for the fixed point function $U^*(\phi^2)$ is as follows. Invert $t(\phi^2)$ to get $\phi^2(t)$, and write

$$\phi^2 = \sum_{m=1}^{\infty} W_m(\mu) t^{m-1} \,. \tag{3.50a}$$

Also solve $t'(\phi^2)$ for ϕ^2 and get

$$\phi^2 = \sum_{m=1}^{\infty} W_m(R_b\mu) t'^{m-1} \,. \tag{3.50b}$$

The renormalization group transformation can then be written

$$W_m(R_b\mu) = [W_m(\mu) - W_m^*] b^{d-2m} + W_m^* \,, \tag{3.51a}$$

where

$$W_m^* = (-1)^{m+1} n K_d \Lambda^{d-2m} (d-2m)^{-1} \,, \tag{3.51b}$$

and K_d is given by eq. (3.39b).

The renormalization group transformation is most elegant when written in terms of "scaling fields" $g_m(\mu)$. These scaling fields [3.10] are the generalization of the eigenvectors v_m of eq. (3.34), and obey the equation

$$g_m(R_b\mu) = g_m(\mu) b^{y_m} \,, \tag{3.52a}$$

where the eigenvalues y_m are

$$y_m = d - 2m \,. \tag{3.52b}$$

The $g_m(\mu)$ are given in terms of the $W_m(\mu)$ by

$$g_m(\mu) = W_m(\mu) - W_m^* \,. \tag{3.52c}$$

The fixed point function $t^*(\phi^2) = 2\,\partial U^*(\phi^2)/\partial\phi^2$ is given implicitly by

$$\frac{\phi^2}{b_1^*} = 1 - (d-2)\Lambda^{2-d} \int_{\Lambda}^{\infty} \mathrm{d}p\, p^{d-1}[(t^* + p^2)^{-1} - p^{-2}]\,. \tag{3.53}$$

The solution of eq. (3.53) cannot be expressed in general as a closed analytic form, but some properties of the solution are immediate:

(i) $U^*(\phi^2)$ is a minimum at $\phi^2 = b_1^*$ for $2 < d < 4$. Near the minimum

$$U^*(\phi^2) \approx U^*(b_1^*) + (\phi^2 - b_1^*)^2/2b_2^*\,. \tag{3.54a}$$

There are no other minima and $\mathrm{d}t^*/\mathrm{d}\phi^2 > 0$ always.

(ii) For $d \geq 4$, $U^*(\phi^2) = t^*(\phi^2) = 0$, i.e., only a trivial (*free-field*) fixed point exists.

(iii) For small $\varepsilon = 4 - d > 0$, the solution is

$$U^*(\phi^2) = \frac{8\pi^2\varepsilon}{n}(\phi^2 - n\Lambda^2/32\pi^2)^2\,. \tag{3.54b}$$

For $2 < d < 4$, the exponent $y_1 = d - 2$ is positive (relevant), and all the other y_m are negative (irrelevant). Thus the anomalous dimension γ_{ϕ^2} of the operator ϕ^2 is given by

$$\gamma_{\phi^2} = 2 - y_1 = \varepsilon\,, \tag{3.54c}$$

and vanishes as $d \to 4$.

The above analysis is particularly instructive because it shows explicitly that the form of the action is not invariant in general under blocking transformations. Moreover, the precise way in which the action is altered under such a transformation depends on the details of the scheme chosen. For example, the scaling fields $g_m(\mu)$ change in a way [cf. eq. (3.52a)] which depends upon b. Thus, the μ [and therefore $U(\phi^2)$] vary in a manner which is determined by the blocking transformation. However, the y_m do *not* depend upon b. This is a rather important point, and is worthy of elaboration.

When a renormalization group transformation is performed, the original action is mapped into a new action, which can be considerably different in its functional form. Two important properties should be preserved however. These are (i) the spacetime dimensionality of the system, and (ii) the symmetry group of the order parameters which distinguish the phases of the theory. Thus, as the renormalization group transformation is repeated, the original system is mapped through a series of new systems, which typically share only these common properties. It is expected that the number of fixed points of the theory and their associated critical exponents will remain the same. (In particular, the number of relevant directions at a fixed point should not depend upon the details of the blocking transformation.) This idea is the so-called "universality hypothesis" [3.11], which says that statistical systems that share properties (i) and (ii) should also demonstrate the same critical behavior. It is similar to the idea that the infinite-cutoff limit of a field theory should be independent of the definition of the cutoff. This contemporary picture of renormalization [1.51] (and the meaning of renormalizability) is reviewed in ref. [3.12]. The basic idea is as follows: begin with a field theory defined via an action and fixed momentum cutoff. This action is totally arbitrary, and can contain terms which a power-counting analysis would indicate are "unrenormalizable". Next systematically integrate out the high-momentum components in this action; at a momentum scale much lower than the cutoff it will be found that the "unrenormalizable" terms in the original action can be accounted for by a redefinition of the renormalized coupling constants at the lower sale. Specifically, the unrenormalizable interactions correspond to "irrelevant" directions in the flow space of coupling constants as the cutoff is decreased.

The fact that the number of renormalized coupling constants in a field theory equals the number of "relevant" directions (or directions of instability) at a fixed point is well known [1.51, 3.6]. The reason is that as the infinite-cutoff limit is taken, it is necessary to adjust as many renormalized parameters as there are relevant directions in order to reach the critical surface. A more heuristic explanation is to say that as this limit is approached, the trajectories are followed in a backward direction, and so the critical surface has as many degrees of instability as there are renormalized coupling constants. *Marginal* directions at the fixed point imply that in only one region of coupling constant space another free renormalized parameter exists; thus there should be a *bound* on the value of this parameter (see section 5).

The accepted picture of how this comes about can be adumbrated in the fashion of Wilson and Kogut [1.51] for the case of the simple cutoff ϕ^4 theory eq. (3.1). Consider fig. 3.2. The action $S_0(\Lambda_0)$ can be defined in terms of the dimensionless parameters

$$r_0(\Lambda_0) \equiv m_0^2 \Lambda_0^{-2}\,, \qquad u_0(\Lambda_0) \equiv \lambda_0 \Lambda_0^{d-4} \tag{3.55a,b}$$

[cf. eq. (3.2)]. Define $\xi_0(\Lambda_0)$ as the dimensionless correlation length for $S_0(\Lambda_0)$. From the discussion of eq. (3.7) it is clear that $\xi(\infty)$ must be infinite in order that the renormalized mass m_R is finite.

The conventional way to achieve this is to give m_0 a cutoff dependence such that the renormalized mass m_R and coupling constant λ_R are cutoff independent as $\Lambda \to \infty$. This can be accomplished by the following procedure. The dimensionless correlation length $\xi(r_0, u_0)$ can be defined independently of Λ_0. Choose m_R and define

$$\Lambda_0 \equiv m_R \xi(r_0, u_0)\,. \tag{3.56}$$

The curve $S_D(\Lambda_0)$ in fig. 3.2 is generated by fixing the value of u_0 and solving eq. (3.56) for r_0. From eqs. (3.55) the values of m_0^2 and λ_0 are determined. The limit $\Lambda_0 \to \infty$ of infinite cutoff is obtained by

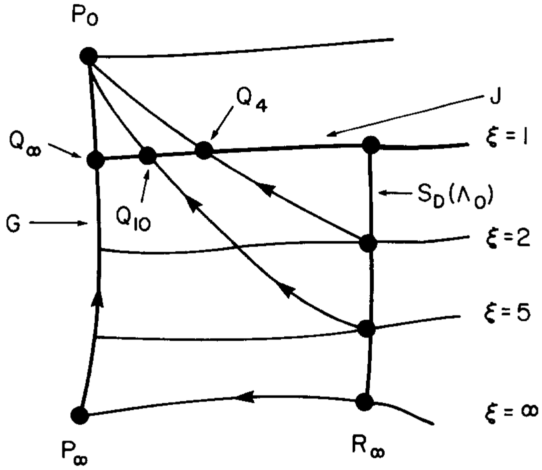

Fig. 3.2. Schematic diagram of the predicted flow structure of four-dimensional ϕ^4 field theory.

letting $r_0(\Lambda_0, u_0)$ approach the critical value $r_{0c}(u_0)$. By eq. (3.56) m_R is finite and independent of the initial u_0.

In fig. 3.2 the curve $S_D(\Lambda_0)$ is indicated schematically. (For reasons of clarity, the plot is two dimensional.) By use of the renormalization group, a set of curves $S_t(\Lambda_0)$ can also be generated. These curves depict the evolution of the action under the renormalization group transformation

$$\Lambda_0 \to \Lambda_t \equiv e^{-t}\Lambda_0 \,. \tag{3.57}$$

The curves $S_t(\Lambda_0)$ can be thought of as defining surfaces of constant "physics".

For a given Λ_0 there is a value of t for which $S_t(\Lambda_0)$ intersects the surface J where ξ equals one. Thus $S_t(4)$ intersects J at a point Q_4, which describes the same physics as the original action $S_D(4)$. Likewise the action at point Q_{10} defines the same system as $S_D(10)$. Note that both Q_4 and Q_{10} define theories with the same renormalized mass (sometimes called the "inverse correlation length in physical units").

As the cutoff Λ_0 increases without bound, the trajectory falls closer and closer to the point Q_∞. When Λ_0 is infinite, the trajectory proceeds from the point R_∞ and ends at the fixed point P_∞. Since P_∞ has one relevant direction ($P_\infty \to P_0$), only one parameter of the theory (m_R) is free; the other (λ_R) is determined by the details of the flow structure. However, at every stage in the process a finite renormalized theory exists. Hence the limit $\Lambda_0 \to \infty$ exists and is independent of u_0.

The above discussion makes several simplifying assumptions. First, additional irrelevant interactions are ignored, though it is easy to see that they do not affect the qualitative details of the argument. It is also assumed that the fixed point P_∞ has only one relevant direction. This is not known, strictly speaking, for four-dimensional ϕ^4 theory, since its triviality has not been proved. However, the above analysis does display much of the machinery of the renormalization group, and how physical concepts are phrased within its framework.

Given the obvious complexity of the renormalization group, it is sensible to ask whether it is superior to direct numerical methods of solution. In fact, as is shown below, approximate analytical and numerical techniques melded with the renormalization group yield a calculational technique of great potential (the Monte Carlo renormalization group). The difficulty of carrying out the renormalization group is much less than that involved in solving the model, because the renormalization group is a transformation of nonsingular parameters and has much more room for approximations.

3.2.4. Approximate momentum-space renormalization group methods

Typically it is not possible to present an exact renormalization group transformation for a given problem. Accordingly, many approximate methods have been developed and applied to ϕ^4 field theory. These can be separated into momentum-space and position-space renormalization group techniques according to the criterion used to partition the degrees of freedom of the system. The results of all of these methods are at least consistent with the idea that ϕ^4 field theory is trivial in four spacetime dimensions. Several of these procedures are reviewed below.

Wilson's approximate recursion formula. The first explicit construction of an approximate momentum-space renormalization group was applied to ϕ^4 field theory by Wilson [3.7]. Although the approximations used in its derivation (see also refs. [1.5], [3.4] and [3.13]) are highly dubious, it is a remarkable achievement, and was instrumental in launching the renormalization group approach to critical phenomena. Even more interesting is the fact that improvements to the original approximations make the end results worse [3.14]. It is evident that approximate treatments of the renormalization group are very complicated (and a bit magical).

The Wilson recursion relation is derived in the above references. The action at iteration number l is

$$S_l = \int \mathrm{d}^d x\,[\tfrac{1}{2}c_l(\partial_\phi)^2 + U_l(\phi)]\,, \tag{3.58}$$

where $l=0$ defines the starting point, in a d-dimensional Euclidean volume Ω. It is assumed that $U_0(\phi) = U_0(-\phi)$. The momentum components in the theory between $\Lambda/2$ and Λ are integrated out (Λ is the momentum cutoff); thus the transformation has scale factor $b=2$. It can be shown that

$$c_{l+1} = c_l \cdot 2^{-\eta}\,. \tag{3.59}$$

Thus a fixed point only occurs for η equal to zero. The recursion relation for the fixed point ($\eta=0$) solution of the potential term $U(\phi)$ can be written in terms of $Q(\phi) \equiv \Omega U(\phi)$ as follows:

$$Q_{l+1}(\phi) = -2^d \ln[I_l(\phi\cdot 2^{1-d/2})/I_l(0)]\,, \tag{3.60a}$$

$$I_l(\phi) = \int \mathrm{d}y\,\exp\{-y^2 - \tfrac{1}{2}[Q_l(\phi+y) + Q_l(\phi-y)]\}\,. \tag{3.60b}$$

Equations (3.60) define a sequence of functions $\{Q_l(\phi)\} = \{Q_0(\phi), Q_1(\phi), \ldots\}$ given an initial function $Q_0(\phi)$.

This recursion relation yields the exact gaussian critical exponent ($\nu = \frac{1}{2}$) in four dimensions, as well as the exact leading terms in an expansion in $\varepsilon = 4-d$. Moreover, given an initial potential of the form

$$U_0(\phi) = a\phi^2 + b\phi^4\,, \tag{3.61}$$

the renormalized potential gradually approaches the trivial gaussian fixed point result $U^*(\phi) = 0$ in four dimensions. Specifically, for a large number l of iterations of the recursion relation, $U_l \propto l^{-1}$.

Since the cutoff in the theory is correspondingly rescaled by 2^l, the approach to trivial free-field behavior is logarithmic in this scale factor. This behavior is also predicted in other formulations of the renormalization group (see ref. [3.15] and section 13 of ref. [1.51]). The recursion relation yields the exact critical exponents for an n-component scalar field theory in the large-n limit for arbitrary d, and also gives good numerical results for $d=2$ and 3 [1.51]. A model has been derived [3.14] for which η is not automatically zero at a fixed point; also "hierarchical" models exist for which the recursion relation is exact [3.16].

The derivation of this recursion relation eqs. (3.59, 60) relies upon "uncontrolled" approximations. It is not the first term in an expansion in any small parameter, and it is not obvious how to improve the calculation. This is a general feature of most approximate renormalization group methods at present. One can thus only judge the validity of such approximations by comparison with the results of other techniques (such as those described below). In this respect it does rather well. Nevertheless it is clear that the present situation is far from optimal.

Renormalization group from high-temperature series. A more accurate renormalization group was developed by Wilson ([3.17] and section 13 of ref. [1.51]) and applied in an unsuccessful search for a second fixed point in four-dimensional pure ϕ^4 field theory. The basic philosophy of this approach was that if a second (and possibly nontrivial) fixed point exists, then the renormalization group flows should slow down as the boundary of the domain of attraction of this fixed point is approached. This idea is illustrated in fig. 3.3. The gaussian fixed point is labelled P_G, and second fixed point is denoted by P_B. The quartic coupling is labelled u, and w defines another coupling which is irrelevant at the gaussian fixed point. If $t_{gate}(u_0)$ is defined to be the time that a given trajectory starting at point u_0 takes to pass the gate, then

$$\phi(u_0) = [dt_{gate}(u_0)/du_0]^{-1} \tag{3.62}$$

vanishes as u_0 approaches the boundary of the domain of the gaussian fixed point. In this sense $\phi(u_0)$ is like the original ψ function of Gell-Mann and Low, or the beta function.

The function $\phi(u_0)$ was evaluated [1.51] from a high-temperature series generated by the method of ref. [3.18]. It was found that the u_0 axis on the critical surface in the space of initial interactions lies *entirely* within the free-field domain. The effect of additional initial interactions (such as ϕ^6, ϕ^8, $\phi^2\nabla\phi^2$, etc.) was not studied. These conclusions are completely in accord with the notion that ϕ^4 is a free field theory in four dimensions.

Functional integral methods in the momentum-space renormalization group. It is also possible to develop exact functional integro-differential equations which express how the action changes in a cutoff field theory when an infinitesimal number of high-momentum components of the fields are integrated out. An example is derived in section 11 of ref. [1.51]. Subsequent work [3.19] based upon this equation was used to prove rigorously that a non-gaussian fixed point exists in three spacetime dimensions, and to bound the values of the corresponding critical indices. Similar equations were derived in ref. [3.20] and used for approximate calculations in ϕ^4 field theory in ref. [3.21]. (No nontrivial fixed points were found in four dimensions, consistent with the picture generated via the perturbative renormalization group.) In this approach the difficulties associated with defining an approximate renormalization group are reduced to the more routine problem of solving a functional differential equation (see also ref. [3.12]). Indeed, the bold assertion has been made that such equations will "probably be the basis for most [future] work on the renormalization group" [1.51]. Nevertheless this approach runs into trouble (as does any momentum-space renormalization group) when gauge theories are included. The reason is that an infinite number of gauge degrees of freedom are associated with each point in momentum space; thus singularities are encountered even when an infinitesimal volume in momentum space is integrated out. This problem can be avoided by fixing the gauge, but then the renormalized action which results can be wildly noninvariant under gauge transformations. The complexity of the problem thus increases enormously. These difficulties in fact originally formed part of the motivation for the development of lattice gauge theories [3.2].

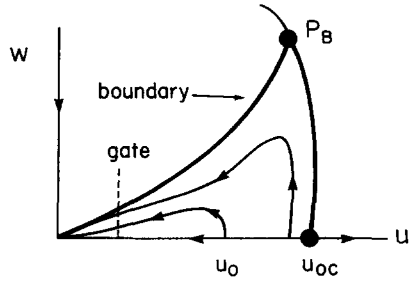

Fig. 3.3. Possible topology of pure ϕ^4 field theory in the presence of a second fixed point P_B.

3.2.5. Approximate real-space renormalization group methods

The above-mentioned difficulties associated with the integration of gauge degrees of freedom can be avoided in the real-space renormalization group (see, for example, refs. [3.22–3.27] and section 5). Specific applications to spin systems are well reviewed in ref. [3.5]; recall that the limit of large quartic coupling in a pure ϕ^4 theory is in fact an Ising spin model. The application of such ideas to ϕ^4 and other quantum field theories is largely unknown territory, but some promising possibilities are explored below.

The Kadanoff variational renormalization group. Impressive success in the calculation of critical exponents was obtained most notably by the Kadanoff variational technique ([3.25], with extensive further work reviewed in ref. [3.26]). Typically critical exponents for spin systems can be obtained with this simple analytical formalism to three-digit accuracy. Moreover, it gives the critical exponents of the gaussian fixed point in ϕ^4 field theory correct to first order in an expansion in ε ($\equiv 4 - d$), and thus is consistent with trivial behavior.

The basic point of this approach is to note that

$$\langle \mathrm{e}^{O} \rangle_S \geq \mathrm{e}^{\langle O \rangle_S} , \qquad (3.63)$$

where expectation values $\langle \cdots \rangle$ are computed via the definition

$$\langle O \rangle_S = Z_S^{-1} \int \mathrm{D}\phi \; O \, \mathrm{e}^{-S} , \quad Z_S \equiv \int \mathrm{D}\phi \, \mathrm{e}^{-S} . \qquad (3.64)$$

For a general field theory the partition function Z_S cannot be calculated exactly. Suppose, however, that the partition function for a related problem with action $S + \Delta S$,

$$Z_{S+\Delta S} \equiv \int \mathrm{D}\phi \, \mathrm{e}^{-(S+\Delta S)} , \qquad (3.65)$$

can be solved exactly. Then, given a definition of the free energy,

$$\mathrm{e}^{-F_S} \equiv Z_S , \qquad (3.66)$$

eq. (3.63) implies that

$$F_{S+\Delta S} \leq F_S , \qquad (3.67)$$

provided that

$$\langle \Delta S \rangle_S = 0 . \qquad (3.68)$$

Equation (3.68) is easily satisfied (by translation invariance) if ΔS is taken as an "interaction moving" operation.

This procedure can be incorporated into a real-space renormalization group scheme which includes variational parameters in the projection operator used to define block fields. Optimization of the transformation then leads to a recursion relation which defines a greatest lower bound on the free energy of the system. Despite an obvious criticism (it is not clear why a good approximate free energy implies good critical exponents), this approach is highly successful.

The Monte Carlo renormalization group. The Monte Carlo renormalization group [3.27–3.30] is a combination of the ideas of the real-space renormalization group with Monte Carlo simulation [3.31]. In principle it is a technique of great power, for it can be systematically improved to arbitrary accuracy. In practice, this combination of numerical analysis with judicious approximation can lead to techniques of some utility in the nonperturbative analysis of quantum field theory.

Operationally the method proceeds as follows. First, a set of original (or "site") configurations is generated by numerical means according to the distribution

$$e^{-S\{\phi\}}\,D\phi\,,$$

where $S\{\phi\}$ is the original action, defined over the space of the fields $\{\phi\}$. Next, block fields $\{\phi\}$ are generated according to a projection operator $P[\{\phi'\},\{\phi\}]$, normalized so that

$$\int D\phi'\,P[\{\phi'\},\{\phi\}]=1\,. \tag{3.69a}$$

The block action $S'\{\phi'\}$ is defined via

$$e^{-S'\{\phi'\}}=\int D\phi'\,P[\{\phi'\},\{\phi\}]\,e^{-S\{\phi\}}\,. \tag{3.69b}$$

[Note that in a real calculation with scale factor b in d dimensions there are b^d site fields for each block field.] Thus the end result is to generate a set of block configurations of the $\{\phi'\}$. These configurations are distributed with the weight (3.69b); it remains to extract the block action $S'\{\phi'\}$ which generates them. Once the block action is extracted, the block coupling constants which define it can be evaluated in terms of the site couplings, and the renormalization group flows constructed.

In the first calculation for ϕ^4 field theory [3.29] configurations were generated according to an action

$$S=-K\sum_{\langle ij\rangle}\phi_i\phi_j+\sum_i U_0(\phi_i)\,,\quad U_0(\phi_i)\equiv\lambda(\phi_i^2-f)^2\,, \tag{3.70}$$

where $\langle ij\rangle$ refers to all nearest-neighbor pairs on a hypercubic lattice, each summed *once*. The system was divided into hypercubic blocks of b^d fields, and the block field ϕ' for each block was defined as

$$\phi'=\operatorname{sign}\Big(\sum_{\text{block}}\phi_i\Big)\Big(b^{-d}\sum_{\text{block}}\phi_i^2\Big)^{1/2}\,, \tag{3.71}$$

in terms of the b^d site fields $\{\phi\}$ within the block. In the Ising model limit of large λ the block spin is thus assigned according to the popular "majority rule" prescription [3.5]. This rule defines the block Ising spin as the sign of the sum of the spins in the block, i.e., as the same value as the "majority" of spins.

It is possible [3.29] to fix all but one of the block fields in such a fashion as to eliminate most interactions in the blocked system. The remaining block field ϕ_0' then fluctuates according to a probability distribution

$$P(\phi_0)\propto\exp[-U(\phi_0')]\,, \tag{3.72}$$

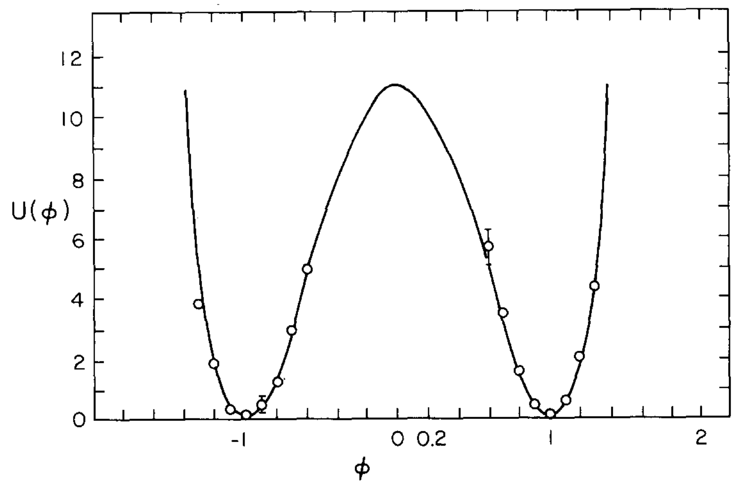

Fig. 3.4. Block renormalized potential $U(\phi)$ plotted versus ϕ, taken from ref. [3.29]. Data points are extracted (eq. 3.72) by direct measurement, while the solid line denotes the curve obtained by evaluation of moments (eq. 3.74).

from which the potential $U(\phi_0')$ can be extracted. This potential is well approximated by a form

$$\tilde{U}(\phi_0') = \lambda'[(\phi_0')^2 - f']^2 , \tag{3.73}$$

where the block couplings λ' and f' are determined [3.39] from expectation values derived from the moment equations,

$$\int_{-\infty}^{\infty} d\phi_0' \frac{d}{d\phi_0'} (\phi_0'^n e^{-U(\phi_0')}) = 0 , \quad n = 1, 2 . \tag{3.74}$$

In fig. 3.4 $U(\phi_0')$ is plotted (data points, eq. 3.72) versus the moment-extracted form (solid line eqs. 3.73, 74). Thus it can be seen that with a judicious choice of the blocking procedure, the block action can be approximated by a simple form from which the renormalized couplings can be extracted. Subsequent calculations with this and other blocking procedures verify this result [3.32]. A flow diagram for the theory can thus be obtained, and fixed points located. Critical exponents can also be extracted, in principle to arbitrary accuracy. Again, all results are consistent with triviality in pure ϕ^4 theory.

3.2.6. Rigorous renormalization group methods

Progress has also been made in the development of rigorous approaches to the renormalization group. One seminal approach is known as "phase-cell localization" [3.33]. The basic idea of this approach is to partition the degrees of freedom of a given system into sets which are finite and independent. These general ideas have been used by many authors [3.34], and have played a part in early rigorous demonstrations such as the linear lower bound on the vacuum energy of $P(\phi)_2$ models [3.35], the renormalization of ϕ^4 field theory in three dimensions [3.33], and in the structure of estimates in the cluster expansion [3.36]. More recent results include the construction of the two-dimensional Gross–Neveu model [3.37, 3.38]. Other important work was done by Gallavotti and collaborators [3.39] and by others (see ref. [3.40] for further references).

One general conclusion of this line of thought is that perturbative renormalizability is neither necessary nor sufficient for the existence of a quantum field theory. For instance, the (perturbatively nonrenormalizable) Gross–Neveu model in $2+\varepsilon$ dimensions can in fact be constructed [3.41]. This model is defined by replacing the free fermionic propagator $-(p\cdot\gamma)^{-1}$ by $-|p|^{\varepsilon}(p\cdot\gamma)^{-1}$. The theory lacks Osterwalder–Schrader reflection positivity, however. A similar approach can be used to construct $\lambda\phi^4$ field theory with negative coupling ($\lambda<0$) [3.42]. This model is metastable and thus also lacks reflection positivity. (Other efforts to escape triviality by nonstandard formulations of ϕ^4 field theory are covered briefly in the next section.) Constructions of this form do, however, show that the verdict on whether or not a theory is trivial should be based upon more information than the meager knowledge gained from perturbation theory. Further interesting results are reviewed in refs. [3.43] and [3.44].

Finally, it is interesting to mention a new result [3.45] produced by a novel approach—a "computer assisted proof". In this paper it is shown that the hierarchical ϕ^4 field theory has a nongaussian fixed point in three dimensions. The method used generates a very large number of inequalities which are subsequently checked and assembled by computer.

3.3. Connections with the historical approach

In the above subsections, the problem of triviality was formulated in terms of the Wilson renormalization group. Nevertheless, the development of this theoretical structure was preceded by the so-called "historical" renormalization group [1.44–1.50, 3.1]. Although some of this "historical" formalism has been applied in the above subsections and especially in section 1, a set of definitions has not been presented. In this subsection the historical renormalization group is outlined, with special emphasis paid to the connections between it and the Wilson approach. In particular, the relations between critical exponents and anomalous dimensions are clarified, and such concepts as fixed points, flow diagrams, and the universality hypothesis are expressed in this formalism.

The analysis begins by studying the theory on the critical surface where the renormalized masses of the theory vanish. Once this surface is mapped out, equations in the whole critical region can be developed.

Another idea related to the renormalization group leads to the so-called Callan–Symanzik equations [3.6]. These are useful for a theory renormalized at μ equal to zero, but with a finite renormalized mass. Here the comparison is between theories with different values of the renormalized mass. Unfortunately the zero-mass, $m_R\to 0$, limit is difficult to express in this framework. Because of this technical problem the Callan–Symanzik equations are slowly being displaced in favor of an expansion of the renormalization group equations around the critical (zero mass) region. Interested readers can find discussions of the Callan–Symanzik equations in refs. [1.50] and [3.1]; however, for reasons of space they are not considered further here.

Finally, using an approach due originally to Zinn-Justin [3.47, 1.50], and closest in spirit to the above ideas of Wilson, differential equations relating different bare theories with the same renormalized coupling constants are derived below.

Because of the fact that this review is intended primarily for high-energy physicists (to whom the historical renormalization group is well known), this subsection is limited to the exploration of parallels between the Wilson and other renormalization group schemes. Readers interested in further details may find refs. [3.1] and [1.50] useful.

3.3.1. Conventions and notation

It is convenient [1.50] to define a scalar field theory in terms of the complete Euclidean Green's functions, or Schwinger functions. These are defined via a Euclidean functional integral,

$$Z(J) = K^{-1} \int \mathrm{D}\phi \, \exp\left(-S(\phi) + \int \mathrm{d}^d x \, J(x)\phi(x)\right), \tag{3.75a}$$

where K is chosen so that

$$Z(0) = 1 . \tag{3.75b}$$

The Schwinger functions $G^{(N)}(x_1, \ldots, x_N)$ are defined implicitly by a power series expansion of eqs. (3.75) in $J(x)$,

$$Z(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, J(x_1) \cdots J(x_N) G^{(N)}(x_1, \ldots, x_N) . \tag{3.76}$$

The connected Green's functions $G_{\mathrm{c}}^{(N)}(x_1, \ldots, x_N)$ are defined by a generating function,

$$\begin{aligned} W(J) &= \ln Z(J) \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, J(x_1) \cdots J(x_N) G_{\mathrm{c}}^{(N)}(x_1, \ldots, x_N) . \end{aligned} \tag{3.77}$$

The historical renormalization group can be defined in terms of the one-particle irreducible (OPI) vertex functions, $\Gamma^{(N)}(x_1, \ldots, x_N)$. These are constructed by a Legendre transformation to a new variable $\Phi(x)$,

$$\Phi(x) = \delta W(J)/\delta J(x) = \langle \phi(x) \rangle . \tag{3.78}$$

The generating function $\Gamma(\Phi)$ for the $\Phi^{(N)}(x_1, \ldots, x_N)$ is the Legendre transform of $W(J)$,

$$\Gamma(\Phi) = \int \mathrm{d}^d x \, J(x)\Phi(x) - W(J) \tag{3.79a}$$

$$= \sum_{N=2}^{\infty} \frac{1}{N!} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, [\Phi(x_1) - G^{(1)}(x_1)] \cdots [\Phi(x_N) - G^{(1)}(x_N)] \Gamma^{(N)}(x_1, \ldots, x_N) . \tag{3.79b}$$

The Fourier transform of $\Gamma^{(N)}(x_1, \ldots, x_N)$ is defined via

$$\begin{aligned} &\delta^{(d)}(p_1 + \cdots + p_N) \Gamma^{(N)}(p_1, \ldots, p_N) \\ &\quad = (2\pi)^{-d(N-1)} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, \exp\left(i \sum_{j=1}^{n} x_j p_j\right) \Gamma^{(N)}(x_1, \ldots, x_N) . \end{aligned} \tag{3.80}$$

Some specific examples of the Green's and vertex functions are

$$G_{\mathrm{c}}^{(2)}(k_1, k_2) = \langle \phi(k_1)\phi(k_2) \rangle - \langle \phi(k_1) \rangle \langle \phi(k_2) \rangle = G^{(2)}(k_1, k_2) - G^{(1)}(k_1) G^{(1)}(k_2) . \tag{3.81a}$$

Translation invariance implies that

$$G_c^{(2)}(k_1, k_2) = (2\pi)^d \delta(k_1 + k_2) G_c^{(2)}(k_1)\,, \qquad \Gamma^{(2)}(k_1, k_2) = (2\pi)^d \delta(k_1 + k_2) \Gamma^{(2)}(k_1)\,, \tag{3.81b}$$

where

$$\Gamma^{(2)}(k) = [G_c^{(2)}(k)]^{-1}\,. \tag{3.81c}$$

Physical particles are defined by poles in $G_c^{(2)}(k)$, and thus *zeroes* of $\Gamma^{(2)}(k)$.

It is also useful to define $\phi^2(x)$ insertions via a source $t(x)$ for this composite field,

$$S \to S - \int d^d x\, J(x)\phi(x) + \frac{1}{2}\int d^d x\, t(x)\phi^2(x)\,. \tag{3.82}$$

The generating functional techniques outlined above can then be used to define OPI vertex functions $\Gamma^{(L,N)}(q_1, \dots, q_L; p_1, \dots, p_N)$ for L insertions of the operator $\phi(x)$ and N insertions of $\phi^2(x)$ [3.48, 1.50].

3.3.2. *Renormalization group equations for massless ϕ^4 theory*

The process of renormalization is well familiar to field theorists. The basic idea is that, whereas the so-called "bare" vertex and Green's function defined above diverge in the limit of large cutoff, it is possible to define finite renormalized functions. In order to do so it is also necessary to define renormalized masses and couplings for the theory.

The $\Gamma^{(N,L)}$ for (N, L) not equal to $(0,2)$ or $(0,0)$ are multiplicatively renormalizable, at least in perturbation theory. This means that new renormalized vertex functions $\Gamma_R^{(N,L)}(q_i, p_i)$ can be defined,

$$\Gamma^{(N,L)} = (Z_\phi)^{N/2}(Z_{\phi^2})^L \Gamma^{(N,L)}\,, \quad (N, L) \neq (0,2) \text{ or } (0,0)\,, \tag{3.83}$$

where the Z_ϕ or Z_{ϕ^2} are field renormalization constants determined by conditions described below. The vertex function $\Gamma_R^{(0,2)}$ is *additively* renormalizable,

$$\Gamma_R^{(0,2)} = (Z_{\phi^2})^2[\Gamma^{(0,2)} - \bar{\Gamma}^{(0,2)}]\,, \tag{3.84}$$

and is finite when $\bar{\Gamma}^{(0,2)}$ is chosen appropriately. A similar result holds for $\Gamma^{(0,0)}$.

The renormalized vertex functions $\Gamma_R^{(N,L)}$ are written in terms of renormalized masses m_R and couplings λ_R. Thus four constraints are needed to define four renormalizations m_R, λ_R, Z_ϕ, and Z_{ϕ^2}. These constraints (or "renormalization conditions") are to a large extent arbitrary. Physical results should be independent of the renormalization group scheme used. It is this independence which, in part, implies the renormalization group.

The following adumbration of the historical renormalization group follows the work of Amit [3.1]; also see refs. [3.49] and [1.50]. A convenient choice of the renormalization scheme is given by the equations

$$m_R \neq 0\colon \quad \Gamma_R^{(2)}(q_i = 0) = m_R^2\,, \tag{3.85a}$$

$$(\partial/\partial q^2)\Gamma_R^{(2)}(q_i = 0) = 1\,, \tag{3.85b}$$

$$\Gamma_{\mathrm{R}}^{(4)}(q_i = 0) = \lambda_{\mathrm{R}} , \tag{3.85c}$$

$$\Gamma_{\mathrm{R}}^{(2,1)}(q_i = 0) = 1 . \tag{3.85d}$$

Additional renormalization conditions like

$$\Gamma_{\mathrm{R}}^{(0,2)}(p_i = 0) = 0 \tag{3.85e}$$

can be used to fix $\bar{\Gamma}^{(0,2)}$.

The renormalization conditions eqs. (3.85) suffice when the renormalized mass m_{R} is finite. If m_{R} vanishes (as happens at the critical point when the cutoff is finite) the required renormalization constants develop infrared divergences. Thus it is more appropriate to employ a different scheme for the massless theory. An arbitrary renormalization point μ^2 is traditionally introduced as follows:

$$m_{\mathrm{R}}^2 = 0: \quad \Gamma_{\mathrm{R}}^{(2)}(q_i = 0) = 0 , \tag{3.86a}$$

$$(\partial/\partial q^2)\Gamma_{\mathrm{R}}^{(2)}(q_i)|_{q^2=\mu^2} = 1 , \tag{3.86b}$$

$$\Gamma_{\mathrm{R}}^{(4)}(q_i)|_{\mathrm{SP}} = \lambda_{\mathrm{R}}(\mu) , \tag{3.86c}$$

$$\Gamma_{\mathrm{R}}^{(2,1)}(q_1, q_2, p)|_{\overline{\mathrm{SP}}} = 1 , \tag{3.86d}$$

$$\Gamma_{\mathrm{R}}^{(0,2)}(p)|_{p^2=\mu^2} = 0 , \tag{3.86e}$$

where "SP" means

$$\mathrm{SP}: \quad q_i^2 = \tfrac{3}{4}\mu^2 , \qquad q_i \cdot q_j = \tfrac{1}{4}\mu^2(4\delta_{ij} - 1) ,$$

i.e.,

$$(q_i + q_j)^2 = \mu^2 \quad \text{for } i \neq j , \tag{3.87}$$

and "$\overline{\mathrm{SP}}$" means

$$\overline{\mathrm{SP}}: \quad q_i^2 = \tfrac{3}{4}\mu^2 , \qquad q_i \cdot q_2 = -\tfrac{1}{4}\mu^2 ,$$
$$p^2 = (q_1 + q_2)^2 = \mu^2 . \tag{3.88}$$

Equation (3.86a) is usually chosen to apply at zero momentum, though it is also satisfactory at $q^2 = \mu^2$. In order to satisfy this equation, the bare mass m_0 must be adjusted so that the renormalized mass vanishes [cf. eq. (3.85a)]. Thus $m_0 = m_0(\lambda_0, \Lambda)$ and is not an independent parameter in the critical theory. Note, however, that m_0 is in general nonzero, even though the renormalized mass vanishes.

The field strength renormalization constants Z_ϕ and Z_{ϕ^2} depend explicitly upon μ, Λ, and $\lambda_{\mathrm{R}}(\mu)$,

$$Z_\phi = Z_\phi(\mu/\Lambda, \mu^{-\varepsilon}\lambda_{\mathrm{R}}(\mu)) , \qquad Z_{\phi^2} = Z_{\phi^2}(\mu/\Lambda, \mu^{-\varepsilon}\lambda_{\mathrm{R}}(\mu)) , \quad \varepsilon \equiv 4 - d . \tag{3.89}$$

The bare theory is defined without reference to the scale μ^2. Thus, in the limit $\Lambda \to \infty$ [cf. eq. (3.83)]

$$(\mu\, \partial/\partial\mu)_{\lambda_0, \Lambda}(Z^{-N/2}\Gamma_{\mathrm{R}}^{(N,0)}) = 0 , \tag{3.90}$$

which implies that

$$[\mu\, \partial/\partial\mu + \beta(\lambda_{\mathrm{R}}, \mu)\, \partial/\partial\lambda_{\mathrm{R}} - \tfrac{1}{2}N\gamma_\phi(\lambda_{\mathrm{R}}, \mu)]\Gamma_{\mathrm{R}}^{(N,0)}(q_i; \lambda_{\mathrm{R}}, \mu) = 0\,, \tag{3.91a}$$

in which

$$\beta(\lambda_{\mathrm{R}}, \mu) = (\mu\, (\partial/\partial\mu)\lambda_{\mathrm{R}}(\mu))_{\lambda_0, \Lambda}\,, \qquad \gamma_\phi(\lambda_{\mathrm{R}}, \mu) = (\mu\, (\partial/\partial\mu) \ln Z_\phi)_{\lambda,\Lambda}\,. \tag{3.91b}$$

Both β and γ_ϕ are finite when $\Lambda \to \infty$ in $d = 4$. The functions appearing in eqs. (3.91) represent the results of the $\Lambda \to \infty$ limit.

A renormalization group equation can also be derived for $\Gamma^{(N,L)}$,

$$[\mu\, \partial/\partial\mu + \beta(\lambda_{\mathrm{R}}, \mu) - \tfrac{1}{2}N\gamma_\phi(\lambda_{\mathrm{R}}, \mu) + L\gamma_{\phi^2}(\lambda_{\mathrm{R}}, \mu)]\Gamma_{\mathrm{R}}^{(N,L)}(q_i, p_i; \lambda_{\mathrm{R}}, \mu) = 0\,, \tag{3.92}$$

where

$$\gamma_{\phi^2}(\lambda_{\mathrm{R}}, \mu) = (-\mu\, (\partial/\partial\mu) \ln(Z_{\phi^2}))_{\lambda_0}\,. \tag{3.93}$$

In four dimensions β, γ_ϕ, and γ_{ϕ^2} are dimensionless, and so

$$\beta = \beta(\lambda)\,, \quad \gamma_\phi = \gamma_\phi(\lambda)\,, \quad \gamma_{\phi^2} = \gamma_{\phi^2}(\lambda)\,. \tag{3.94}$$

As always, eq. (3.93) does not include the special cases $(N,L) = (0,0)$ or $(0,2)$. As was pointed out by Coleman [3.50], eqs. (3.91) and (3.92) have an interesting pictorial analogy. The renormalization group equations can be thought of as describing the flow of bacteria in a one-dimensional channel. Time is measured by $\ln \mu$, the instantaneous velocity is $\beta(\lambda)$, and the anomalous dimensions γ_ϕ and γ_{ϕ^2} are (depending upon their signs) source or sink terms.

The renormalization group equation (3.91) can be used to determine the high-momentum behaviour of the vertex function. For example, dimensional analysis yields the result

$$\Gamma_{\mathrm{R}}^{(N,0)}(\rho q_i, \lambda_{\mathrm{R}}, \mu) = \rho^{N+d-1/2}\Gamma_{\mathrm{R}}^{(N,0)}(q_i, \lambda_{\mathrm{R}}, \mu/\rho)\,. \tag{3.95}$$

The combination of eqs. (3.92) and (3.95) then implies that

$$\Gamma_{\mathrm{R}}^{(N,0)}(\rho q_i, \lambda_{\mathrm{R}}, \mu) = \rho^{N+d-Nd/2} \exp\Bigg(-\frac{N}{2}\int_{\lambda_{\mathrm{R}}}^{\bar\lambda(\rho)} \frac{\gamma_\phi(\lambda)}{\cancel{\beta(\lambda)}}\,\mathrm{d}\lambda\Bigg)\Gamma_{\mathrm{R}}^{(N,0)}(q_i, \bar\lambda(\rho), \mu)\,, \tag{3.96a}$$

where

$$s = \ln \rho = \int_{\lambda_{\mathrm{R}}}^{\bar\lambda(t)} \frac{\mathrm{d}\lambda}{\beta(\lambda)}\,, \qquad \partial\lambda(s)/\partial s = \beta(\bar\lambda)\,. \tag{3.96b,c}$$

As pointed out in section 1, the existence of a nontrivial theory is related to the presence of zeroes in the beta function,

$$\beta(\lambda^*) = 0 . \tag{3.97}$$

The values of λ^* which satisfy eq. (3.97) are known as fixed points. At a fixed point, the renormalization group predicts a simple scaling behavior for the vertex functions. Equations (3.91) reduce to

$$[\mu\partial/\partial\mu - \tfrac{1}{2}N\gamma_\phi(\lambda^*)]\Gamma^{(N,0)}(q_i, \lambda^*, \mu) = 0 . \tag{3.98}$$

Equation (3.98) implies that under a rescaling of momenta $q_i \to \rho q_i$,

$$\Gamma_{\mathrm{R}}^{(N,0)} \to \Gamma_{\mathrm{R}}^{(N,0)}(\rho q_i, \lambda^*, \mu) = \rho^{(N+d-1/2)-N\gamma_\phi(\lambda^*)/2}\Gamma_{\mathrm{R}}^{(N,0)}(q_i, \lambda^*, \mu) , \tag{3.99}$$

and in particular [cf. eq. (3.25)]

$$\Gamma_{\mathrm{R}}^{(2,0)}(\rho q, \lambda^*, \mu) = \rho^{2-\gamma_\phi^*}\Gamma_{\mathrm{R}}^{(2,0)}(q, \lambda^*, \mu) . \tag{3.100}$$

Thus $\Gamma_{\mathrm{R}}^{(N,0)}$ behaves as if each of the fields $\phi(x)$ scaled like

$$\phi(x) \to \rho^{d_\phi}\phi(x/\rho) \tag{3.101}$$

at a fixed point, where

$$d_\phi \equiv d/2 - 1 + \eta/2 , \qquad \eta \equiv \gamma_\phi(\lambda^*) \equiv \gamma_\phi^* \tag{3.102}$$

[cf. eqs (3.15)]. That is, each of the fields $\phi(x)$ scales as if it had an additional "anomalous dimension" $\eta/2$ along with its "engineering dimension" $d/2 - 1$ (see ref. [1.51]). Powers of μ are included in order to make the dimensionality of the vertex functions come out right. Thus, at a fixed point

$$\Gamma_{\mathrm{R}}^{(2)}(q_i, \lambda^*, \mu) = C\mu^\eta q^{2-\eta} , \tag{3.103}$$

where C is a constant. Similarly, for the bare functions

$$\Gamma^{(2)}(q, \lambda_0, \Lambda) = C'\Lambda^\eta q^{2-\eta} , \tag{3.104}$$

implying that

$$Z_\phi(\lambda^*, \mu/\Lambda) = C''(\mu/\Lambda)^\eta . \tag{3.105}$$

It is important to point out that the derivation of eq. (3.94) implicitly assumes that $\beta(\lambda)$ has only a single zero at λ^*. If $\beta(\lambda)$ has a double zero at λ^* (as happens for ϕ^4 theory at $\lambda = 0$), then near λ^*

$$\beta(\lambda) = a(\lambda - \lambda^*)^2 , \qquad \gamma_\phi(\lambda) = \gamma_\phi(\lambda^*) + \gamma_\phi'(\lambda^*)(\lambda - \lambda^*) + \cdots . \tag{3.106a,b}$$

The prefactor in eq. (3.96a) is

$$\exp\Bigl(-\frac{N}{2}\int_{\lambda_{\mathrm{R}}}^{\bar\lambda(s)} \frac{\gamma_\phi}{\beta}\,\mathrm{d}\lambda\Bigr) \sim \rho^{-N\gamma_\phi^*/2}(\ln\rho)^{-N\gamma_\phi'/2a} . \tag{3.107}$$

Thus *logarithmic scaling violations* result from the presence of the double zero in $\beta(\lambda)$. This double zero is connected to the fact that λ is a marginal coupling near the gaussian fixed point [cf. eq. (3.42) et seq.]. A further discussion of the implications of marginal operators for scaling violations appears in section 5.4.

3.3.3. Renormalization group equations in the critical region: The relation between ν and γ_{ϕ^2}

In the above subsection, the renormalization group was formulated for the massless ϕ^4 theory. Specifically the bare mass m_0 of the theory was adjusted to a certain value m_c so that the renormalized mass m_R was zero. The massless theory defines (at finite cutoff Λ) a critical theory where the dimensional correlation length ξ is infinite.

In order to understand further the connections between the Wilson and historical renormalization group approaches, it is necessary to consider the theory away from the critical point. The bare Lagrange density has a term

$$m_0^2\phi^2 \equiv m_c^2\phi^2 + (m_0^2 - m_c^2)\phi^2 \equiv m_c^2\phi^2 + \delta m^2\, \phi^2\,. \tag{3.108}$$

The difference δm^2 should be interpreted as the limit of a spatially varying quantity as it tends towards a constant. When δm^2 vanishes, the renormalized mass vanishes as well,

$$\xi \sim m_R^{-1} \sim (\delta m^2)^{-\nu} \tag{3.109}$$

[cf. eq. (3.11)]. Equation (3.109) describes the way that the dimensionless correlation length ξ vanishes at a critical point. The analogy with critical phenomena can be recovered via the identification

$$\frac{\delta m^2}{m_c^2} = \frac{T - T_c}{T_c}\,, \tag{3.110}$$

where T is the "temperature" of the analogous statistical system, and T_c is its critical temperature. The fact that ξ is no longer infinite when δm^2 is finite is implied by the fact that (*vide supra*) δm^2 is a "relevant" coupling at the critical point in the Wilson approach.

Define a new (position independent) variable t by [cf. eq. (3.82)]

$$t \equiv \delta m^2/Z_{\phi^2}\,. \tag{3.111}$$

Then it can be shown that [1.50, 3.1]

$$[\mu\, \partial/\partial\mu + \beta(\lambda)\, \partial/\partial\lambda - \tfrac{1}{2}N\gamma_\phi(\lambda) + \gamma_{\phi^2}\, t\, \partial/\partial t]\Gamma_R^{(N,0)} = 0\,. \tag{3.112}$$

Define

$$\theta \equiv -\gamma_{\phi^2}(\lambda^*)\,. \tag{3.113}$$

Then at a fixed point λ^*

$$[\mu\, \partial/\partial\mu - \tfrac{1}{2}N\eta - \theta t\, \partial/\partial t]\Gamma_R^{(N,0)} = 0\,, \tag{3.114}$$

which has a solution

$$\Gamma_{\mathrm{R}}^{(N,0)}(q_i, t, \mu) = \mu^{N\eta/2} F^{(N)}(q_i, \mu t^{1/\theta}) . \tag{3.115}$$

From dimensional analysis

$$\Gamma_{\mathrm{R}}^{(N,0)}(q_i, t, \mu) = \rho^{N+d-Nd/2} \Gamma_{\mathrm{R}}^{(N,0)}(\rho^{-1} q_i, \rho^{-2} t, \rho^{-1} \mu) . \tag{3.116}$$

Choose

$$\rho = \mu(t/\mu^2)^{1/(\theta+2)} . \tag{3.117}$$

Then

$$\Gamma_{\mathrm{R}}^{(N)}(q_i, t, \mu) = \mu^{x_1}(t/\mu^2)^{x_2} F^{(N)}[q_i \mu^{-1}(t/\mu^2)^{x_3}] , \tag{3.118a}$$

where

$$x_1 \equiv d + N(2-d)/2 , \quad x_2 \equiv \frac{d + N(2-d-\eta)/2}{\theta+2} , \quad x_3 \equiv -1/(\theta+2) . \tag{3.118b}$$

Thus the vertex functions depend only upon the combination $q_i \xi$, with

$$\xi \propto t^{-1(\theta+2)} . \tag{3.119}$$

Equations (3.109) and (3.119) together imply that

$$\nu^{-1} = 2 + \theta = 2 - \gamma_{\phi^2}^{*} , \tag{3.120}$$

which is the desired result.

3.3.4. Renormalization group equations for the bare theory

An approach which was originally suggested by Zinn–Justin [3.7, 1.50] (and falls closest in spirit to the ideas of Wilson) is to consider all bare theories which correspond to a given renormalized theory. Although this implementation of the ideas of the renormalization group is conceptually very different from the historical scheme outlined above, the resulting equations and their solutions are quite similar. Thus we shall only give a brief outline of the method.

Recall that the statement of renormalizability is that the renormalized vertex functions $\Gamma_{\mathrm{R}}^{(N)}$,

$$\Gamma_{\mathrm{R}}^{(N)}(q_i, \lambda_{\mathrm{R}}, \mu) = Z_{\phi}^{N/2}(\lambda_0, \mu/\Lambda) \Gamma^{(N)}(q_i, \lambda_0, \Lambda) , \tag{3.121}$$

have a finite limit which is independent of Λ as $\Lambda \to \infty$. Thus in this limit

$$[\Lambda\,(\partial/\partial\Lambda) \Gamma_{\mathrm{R}}^{(N)}]_{\lambda_{\mathrm{R}},\mu} = 0 \tag{3.122}$$

(for convenience the critical theory with $m_{\mathrm{R}} = 0$ is considered). Equation (3.122) implies that

$$[\Lambda\, \partial/\partial\Lambda + \beta(\lambda_0)\, \partial/\partial\lambda_0 - \tfrac{1}{2}N\gamma_\phi(\lambda_0)]\Gamma^{(N)}(q_i, \lambda_0, \Lambda) = 0\,, \tag{3.123a}$$

where

$$\beta(\lambda_0) = \Lambda(\partial\lambda_0/\partial\Lambda)_{\lambda_R,\mu}\,, \qquad \gamma_\phi(\lambda_0) = -\Lambda(\partial \ln Z_\phi/\partial\Lambda)_{\lambda_R,\mu} \tag{3.123b}$$

are finite [1.50] (at least in perturbation theory) as $\Lambda \to \infty$. Because they are dimensionless, they are also independent of μ and Λ in this limit, and depend only upon λ_0. Equations (3.123) describe the changes in the bare vertex function $\Gamma^{(N)}$ and bare quartic coupling λ_0 required to leave the renormalized theory invariant as the cutoff Λ is varied. They can be used to derive scaling relations like eq. (3.104) directly. Note that in contrast to eq. (3.96c),

$$\Lambda\, \partial\lambda_0(\Lambda)/\partial\Lambda = -\beta(\lambda_0)\,, \tag{3.124}$$

and so here coupling constants flow in a direction opposite to the flow in the historical renormalization group. Thus as $\Lambda \to \infty$ the coupling $\lambda_0(\Lambda)$ flows (at least for small λ_0) toward the gaussian fixed point $\lambda_0 = 0$. The resulting scaling at the gaussian fixed point occurs independently of the initial value of $\lambda_0(\Lambda)$; this is an example of the phenomenon of "universality" alluded to above.

In this last subsection the ideas of fixed points, critical exponents, renormalization group flows, and universality have been shown to be common to both the Wilson and historical approaches. Much of the work done in the Wilson approach has its parallels in the historical formalism as well. For example, Wilson's approximate recursion formula was also derived from a diagrammatic expansion formula by Polyakov [1.51]. It should be evident that the problem of triviality has a natural place in either setting. In section 5, the application of these ideas to problems of direct interest to particle physicists (most notably the standard model of the weak interaction) is discussed. Possible phenomenological implications of the flow structure of the theory—such as a bounded or predictable Higgs mass—are also mentioned.

4. Other approaches—Selected alternative views of triviality

In this section a few of the remaining pursuits of triviality are considered. The first two subsections respectively cover direct numerical simulation and high-temperature series; following these accounts is a survey of various speculations on ways to construct sensible *nontrivial* ϕ^4 field theories.

4.1. Direct numerical simulation

The fact that lattice field theories are amenable to numerical solution (see also sections 3 and 5) implies that triviality can be studied in this fashion. By numerical means it is possible to evaluate the field strength renormalization and the renormalized coupling constants as a function of lattice spacing a in a given finite volume. The continuum field theory then arises as the limit of vanishing lattice spacing is taken [2.1, 4.1, 4.2]. Extrapolation of the renormalized quartic coupling $\lambda_R(a)$ to zero lattice spacing yields a prediction for its value $\lambda_R(0)$ in the continuum. If a nontrivial continuum ϕ^4 theory is to exist, the value of $\lambda_R(0)$ in this limit must be bounded away from zero [4.3, 4.4]. (Rigorous upper bounds on the value of λ_R have been constructed [4.5]; see also section 2.)

A landmark calculation of this form was performed by Freedman, Smolensky, and Weingarten [4.6]. The lattice theory is (modulo a few changes in notation) governed by an action defined upon a d-dimensional hypercubic lattice with N sites on a side,

$$S = \frac{1}{2} \sum_{n,\mu} [\phi(n) - \phi(n+\mu)]^2 + \tfrac{1}{2}(am_0)^2 \sum_n [\phi(n)]^2 + a^{4-d}\lambda_0 \sum_n [\phi(n)]^4 , \tag{4.1}$$

where as usual n denotes a lattice site, μ $(=1, \ldots, d)$ denotes a direction, and (m_0, λ_0) are respectively the bare lattice mass and quartic coupling constant. Here $\phi(n)$ is a dimensionless lattice field related to its continuum counterpart $\phi(x)$ by

$$\phi(x) = a^{1-d/2}\phi(n) . \tag{4.2}$$

Using standard Monte Carlo techniques, the Fourier transform $\tilde{\phi}(q)$ of $\phi(n)$ is computed,

$$\tilde{\phi}(q) = \sum_n \exp(\mathrm{i}aq \cdot n)\, \phi(n) . \tag{4.3}$$

The quantity $\tilde{\phi}(q)$ is evaluated at q equal to zero and at $q = p \equiv 2\pi/Na$. An adequate definition of the renormalized mass is

$$\text{continuum:} \quad m_{\mathrm{R}}^{-2} = -\frac{d}{dq^2} \ln G(q^2)|_{q=0} , \tag{4.4}$$

in the continuum, where $G(q^2)$ is the renormalized two-point function in momentum space. (Note that if gauge fields are present, the definition (4.4) is not gauge invariant, as it does not correspond to the pole of a propagator.) In lattice terminology, eq. (4.4) can be translated as

$$\text{lattice:} \quad m_{\mathrm{R}}^{-2} = [\langle \tilde{\phi}(0)^2 \rangle - \langle |\tilde{\phi}(p)|^2 \rangle][p^2 \langle |\tilde{\phi}(p)|^2 \rangle]^{-1} . \tag{4.5}$$

In the free-field limit $\lambda_0 = 0$, eq. (4.5) reduces to

$$\text{free lattice:} \quad m_{\mathrm{R}}^{-2} = m_0^{-2}[(\pi/N)\sin(\pi/N)]^{-2} . \tag{4.6}$$

A renormalized momentum-dependent *unitless* coupling constant was also defined,

$$\lambda_{\mathrm{R}}(q) = -(Nam_{\mathrm{R}})^d[\langle \tilde{\phi}(0)^2 |\tilde{\phi}(q)|^2 \rangle - 3\langle \tilde{\phi}(0)^2 \rangle \langle |\tilde{\phi}(q)|^2 \rangle] / \langle |\tilde{\phi}(q)|^2 \rangle . \tag{4.7}$$

Specifically the quantities $\lambda_{\mathrm{R}}(0)$ and $\lambda_{\mathrm{R}}(p)$ were evaluated by these authors. The coupling $\lambda_{\mathrm{R}}(0)$ is a truncated, renormalized connected four-point function at zero momentum. The momentum-dependent coupling $\lambda_{\mathrm{R}}(p)$, by contrast, has one external leg at momentum p and another at $-p$. It also differs from $\lambda_{\mathrm{R}}(0)$ by an overall factor of $\langle \tilde{\phi}(0)^2 \rangle / \langle |\tilde{\phi}(p)|^2 \rangle$. A factor of m_{R}^{d-4} was included in eq. (4.7) to ensure that $\lambda_{\mathrm{R}}(q)$ is dimensionless in arbitrary dimension d. To leading order in λ_0, $\lambda_{\mathrm{R}}(0) \simeq 24 m_{\mathrm{R}}^{d-4}\lambda_0$, while in the limit $Na \to \infty$, $\lambda_{\mathrm{R}}(p) \simeq 24 m_{\mathrm{R}}^{d-4}\lambda_0$, also to leading order in λ_0.

According to the philosophy outlined above, the lattice spacing a is sent to zero while the volume $(Na)^d$ is kept fixed. In practical terms this means that a series of measurements are made upon a sequence of lattices with N taking increasingly larger values (so that the lattice spacing a is progressively smaller). For each value of N, the bare mass m_0 is adjusted until the renormalized mass equals a certain value. The value $m_R = 3.63$ (with estimated error of a few percent) was chosen by these authors [4.6]. This choice of m_R was motivated as a compromise between the simultaneous requirements that the volume be large and the lattice spacing be small in units of m_R^{-1}.

Once this tuning of m_0 was achieved, the value of λ_R was evaluated as a function of the bare quartic coupling λ_0. These results are shown in figs. 4.1–4.3 for $d = 3$ and 4. The contrast between the two cases is evident—in three dimensions (fig. 4.1) λ_R appears to approach a nonzero limiting value as the lattice spacing decreases. This is as expected, for (see section 2) in three dimensions ϕ^4 field theory is nontrivial. On the other hand, in four dimensions a trivial theory is anticipated. Consonant with this expectation, $\lambda_R(0)$ and $\lambda_R(p)$ approach zero uniformly in this limit (figs. 4.2 and 4.3).

The optimism generated by the apparent ease of producing these results must be tempered with realistic judgement, however. First, it is well to remember that a complete error analysis of these and other related results [4.7–4.10] is lacking. Errors arise in such calculations from both systematic effects (like the finite volume of the lattice) and statistical effects (critical slowing down). It is therefore wise to repeat Monte Carlo calculations using alternative approaches, such as Langevin [4.8] and microcanonical [4.11] methods, for in these cases the errors are at least different.

Nevertheless it is exciting to consider the possibilities which open up when these numerical computations are taken seriously. For instance, one novel result of ref. [4.7] was the finding that the two-point function was equal to the free-field correlation function. This result was essentially confirmed in ref. [4.8], where it was found that within error the momentum-dependent field strength renormalization $Z(p)$ was equal to one for all measured momenta. These conclusions imply that the renormalized theory is at best very weakly interacting. Simulations in the broken symmetry phase of the theory [4.9] yield results which also are consistent with the triviality of ϕ^4 field theory in four dimensions.

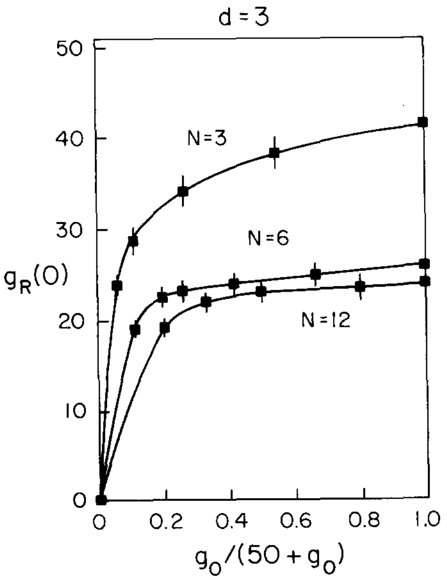

Fig. 4.1. The zero-momentum coupling constant $g_R(0)$ as a function of the bare coupling constant ratio $g_0/(50+g_0)$ for $(\phi^4)_3$ on lattices of size $N = 3$, 6, and 12. Redrawn from ref. [4.6].

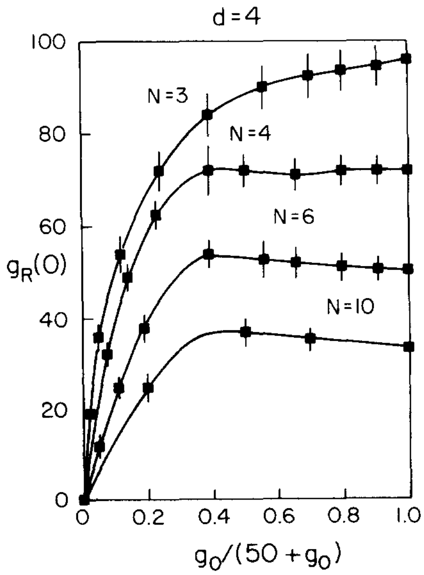

Fig. 4.2. The zero-momentum renormalized coupling constant $g_R(0)$ as a function of the bare coupling constant ratio $g_0/(50+g_0)$ for $(\phi^4)_4$ on lattices of size $N = 3$, 4, 6, and 10. Redrawn from ref. [4.6].

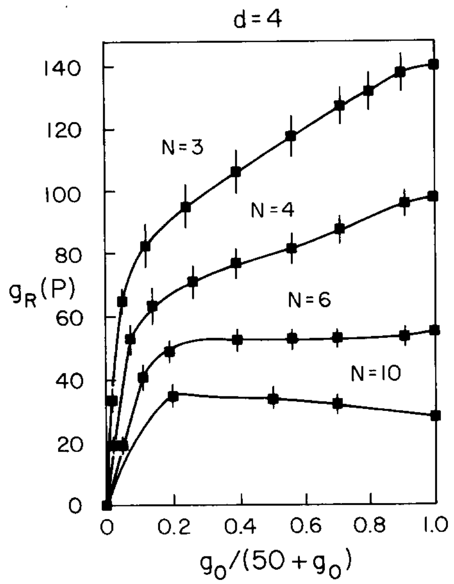

Fig. 4.3. The momentum p renormalized coupling constant $g_R(p)$ as a function of the bare coupling constant ratio $g_0/(50+g_0)$ for $(\phi^4)_4$ on lattices of size $N=3$, 4, 6 and 10. Redrawn from ref. [4.6].

4.2. High-temperature series

The so-called "high-temperature" or "strong-coupling" series approach has been remarkably successful in the calculation of accurate critical exponents and temperatures in statistical mechanics (see, e.g., refs. [4.12, 4.13] for reviews). Much of this work actually concerns the Ising model. However, since (as discussed in section 3) the Ising model is in the same universality class as single-component ϕ^4 field theory, the critical exponents should be the same.

The accuracy of this approach is remarkable. For instance, in an early calculation [4.14], Fisher and Gaunt thus evaluated the free energy and susceptibility for an Ising model on a general d-dimensional hypercubic lattice. These series (in inverse powers of the temperature) were rearranged in a new series in inverse powers of the coordination number q (for a hypercubic lattice, $q=2d$). The critical temperature of this model can thus be developed in a series

$$\frac{kT_c}{qJ} = 1 - q^{-1} - 1\tfrac{1}{3}q^{-2} - 4\tfrac{1}{3}q^{-3} - 21\tfrac{34}{45}q^{-4} - 133\tfrac{14}{15}q^{-5}\cdots. \tag{4.8}$$

Reasonable approximations to the Ising model results are obtained even for $d=2$ or 3 (errors 6% and 1%, respectively) when the series is truncated after the q^{-1} term. Later calculations [4.15] predicted that in four dimensions $\nu = 0.536 \pm 0.003$ (compare the expected mean-field result $\nu=\frac{1}{2}$). Of course this method assumes that the critical exponents vary continuously as a function of spacetime dimensionality d. As is discussed in sections 2 and 3, it is now believed that ν equals $\frac{1}{2}$ when $d>4$, and that only logarithmic corrections to this mean-field result can exist in four dimensions. Logarithmic corrections are easily missed in this approach if due care is not taken, and can lead to inaccurate exponents.

Much of the relevant work with high-temperature series involves the use of so-called "Padé approximants" [4.16, 4.17]. These were first introduced into physics by Baker and collaborators [4.18, 4.19]; excellent reviews can be found in refs. [4.20] and [4.13].

The main ingredient in the Padé recipe (and here only the simplest and most direct approach is discussed) is the approximation of a finite series $F(z)$ by a ratio of two finite polynomials. The $[L, M]$ Padé approximant to $F(z)$ is the ratio of a polynomial $P_L(z)$ of degree L to a polynomial $Q_M(z)$ of degree M,

$$[L, M] \equiv \frac{P_L(z)}{Q_M(z)} \equiv \frac{p_0 + p_1 z + p_2 z^2 + \cdots + p_L z^L}{1 + q_1 z + q_2 z^2 + \cdots + q_M z^M} . \tag{4.9}$$

In eq. (4.9) the coefficients p_i and q_i are chosen so that the series expansion of $[L, M]$ agrees with that of $F(z)$ through order $L + M$, i.e.,

$$F(z) = [L, M] + \mathrm{O}(z^{L+M+1}) . \tag{4.10}$$

The coefficients so chosen are unique [4,20]. The construction of Padé approximants allows one to form an approximate analytic continuation of a given series beyond its radius of convergence and up to (or beyond) a singularity.

In the theory of critical phenomena (and thus in field-theoretic systems) a typical quantity of interest has a singularity structure for $z < z_c$ of the form

$$I(z) \approx (1 - z/z_c)^{-r} J(z) , \tag{4.11}$$

where $J(z)$ is analytic at z_c. Then, if $F(z)$ is taken to be the logarithmic derivative of $I(z)$,

$$F(z) \equiv \frac{\mathrm{d}}{\mathrm{d}z} \ln I(z) \approx \frac{r}{z - z_c} [1 + \mathrm{O}(z - z_c)] \quad \text{as } z \to z_c , \tag{4.12}$$

the Padé approximants to $F(z)$ may yield information on the singularities of $I(z)$. If a given approximant to $F(z)$ has a pole near z_c, then the location of the pole gives an estimate of z_c, while the corresponding residue can be used to estimate r. (The reader should be cautioned at this point that the Padé approximant technology has become a veritable science of its own, and that far more subtle methods than this one do in fact exist [4.16–4.20].)

The results of the Padé analysis are typically displayed as a "Padé table" [4.16]. If $F(z)$ is known as a power series up to z^N, the corresponding Padé table is given by the set of approximants:

$$\begin{array}{lllll} [0,0] & [1,0] & [2,0] & \cdots & [N,0] \\ [0,1] & [1,1] & \cdots & [N-1,1] & \\ \vdots & & & & \\ [0,N] & & & & \end{array} \tag{4.13}$$

Often only the results of the Padé analysis (i.e., the pole location and residue) are tabulated in this form. Usually there are a few "bad eggs", entries in the table which are radically different from the others, but the Padé table normally converges with high accuracy. It should, however, be mentioned that the error analysis for Padé approximants is unfortunately not always an exact science, and a naive approach can mislead the most careful investigator.

Padé techniques allow one to study the behavior of the renormalized quartic coupling λ_R as the critical surface is approached. By analogy with statistical mechanics, this approach to the critical surface can be parameterized by a temperature T, which is assigned a value T_c at the critical surface. In general (see, e.g., refs. [1.51], [1.50], [2.30], and [4.21]), when T is just above the critical value T_c, $\lambda_R(T)$ scales like

$$\lambda_R \sim (T - T_c)^\kappa |\ln(T - T_c)|^P . \tag{4.14}$$

The power κ can be written in terms of standard critical exponents,

$$\kappa = \gamma + d\nu - 2\Delta , \tag{4.15}$$

where γ and ν are respectively the magnetic susceptibility and correlation length indices, and Δ is the "gap" exponent.

The fact [4.5] that λ_R is bounded as $T \to T_c^+$ implies that

$$\kappa \equiv \gamma + d\nu - 2\Delta \geq 0 . \tag{4.16}$$

(The strict equality $\kappa = 0$ is often called a "hyperscaling" relation.) Moreover, this reasoning implies that P must be less than or equal to zero when κ vanishes. (Recall [1.50, 1.51] that the renormalization group predicts $\kappa = 0$, $P = -1$, i.e., that "logarithmic corrections to hyperscaling exist".) Only in the case when κ and P *both* vanish can the theory be nontrivial [2.30]. A particularly impressive series calculation for ϕ^4 field theory was performed by Baker and Kincaid [4.22] (see also refs. [4.23, 4.24]). They obtained high-temperature series up to tenth order and used them to estimate the dimensionless renormalized coupling constant. A scaling law of the form (4.14) was assumed, subject to the (very strong) constraint that P is equal to zero. These authors [4.21] found that $\kappa = 0.30 \pm 0.04$, which is, however, consistent with triviality. For comparison, direct evaluation of high-temperature series for the four-dimensional Ising model (again assuming that P vanishes) yields [4.25] $\kappa = 0.302 \pm 0.0038$.

Other authors have subsequently reexamined these models and have obtained results consistent with the renormalization group prediction of triviality ($\kappa = 0$, $P = -1$) for both pure ϕ^4 field theory [4.26] and the Ising model [4.27] in four dimensions. It is therefore likely that logarithmic corrections to scaling cannot be neglected *a priori* in eq. (4.14), and that the assumption that P vanishes introduces errors in the evaluation of κ.

Triviality has also been studied by the evaluation of the effective potential in a strong coupling series ([4.28], see also ref. [4.29]). Again, the conclusions reached are consistent with the idea that ϕ^4 field theory is trivial in four dimensions.

4.3. *Trying to escape the trap of triviality: The search for loopholes*

The importance of the Higgs mechanism in models of elementary particle interactions has provided a rationale in quests for a nontrivial ϕ^4 theory. It is probably fair to evaluate the present situation by saying that such endeavors generally should still be considered speculations (albeit entertaining ones). Further ideas in this spirit are also elaborated upon in section 5 and 3.2.6.

A remarkable philosophy called *asymptotic safety* has been advanced by Weinberg [4.30] as a way to evade the obloquy often associated with trivial or even nonrenormalizable field theories (like quantum gravity). He points out that in a (cutoff) field theory it is necessary to have a rationale for constructing a unique model with trivial and/or nonrenormalizable interactions. For instance, it is insufficient to argue that nonrenormalizable interactions must be eliminated from a field theory because their presence requires that an infinite number of parameters be evoked. The point is that the attendant logical prodigality can be rectified by, e.g., demanding that all free parameters in the theory equal unity at a given momentum scale.

Rather than imposing such a condition *ex post facto* upon a theory, it is better to have the theory itself produce conditions for consistency. Asymptotic safety gives such a promise—it is simply the requirement that the renormalization group trajectories strike a fixed point in the limit of large cutoff $\Lambda \to \infty$. In all cases considered in ref. [4.30], there are a finite number of "ultraviolet-attractive" eigenvalues at such a fixed point. Therefore, although a cutoff field theory can have an infinite number of parameters (corresponding to trivial and/or nonrenormalizable interactions), the principle of asymptotic safety imposes an infinite number of constraints. A finite number of independent parameters then remains. It is argued that this principle may thus explain or even *replace* renormalizability (and *ipso facto* nontriviality) as a criterion for field theories. See also sections 1 and 5.

Other attempts to construct a nontrivial ϕ^4 theory are often a bit more abstract in nature. Rigorous discussions of triviality (see section 2) often require that a ϕ^4 field theory is defined as an infinite-cutoff limit of a *ferromagnetic* lattice theory. It has been argued [4.31] that this is an asumption whose removal changes the nature of the problem dramatically. Indeed, no argument appears to prevent the existence of an interesting nontrivial ultraviolet limit of an antiferromagnetic lattice ϕ^4 theory, even in $d > 4$. This remains an interesting open problem.

Another possible route for avoiding triviality is the so-called "scale-covariant field theory" [4.32]. The starting point of the approach is the observation that in certain quantum systems the presence of an infinitesimal perturbation can cause a discontinuous change in the eigenfunctions and eigenvalues. Thus the free field solutions are *not* obtained in the limit of a vanishing perturbation (see also ref. [4.33] for a discussion of this phenomenon in path integrals). Instead, a "pseudo-free" solution is reached. It is argued [4.32] that nonasymptotically free theories have these pseudo-free solutions and that perturbative expansions ought to be carried out about these solutions.

An idea which is suggested by the pseudo-free solution to the "independent value model" [4.32] is that it may be appropriate to replace the path integral measure $\Pi_x \, \mathrm{d}\phi(x)$ by

$$\prod_x \frac{\mathrm{d}\phi(x)}{|\phi(x)|^B} \tag{4.17}$$

in the definition of a scalar field theory. When the (classically invisible) parameter B equals one, the measure is locally invariant under the scale transformation $\phi(x) \to s(x)\phi(x)$. For certain values of the parameter B, the $\mathrm{O}(N)$-symmetric ϕ^4 model defined with the measure (4.17) is nontrivial [4.34]. Early numerical work [4.32] suggested that scale-covariant ϕ^4 theory is trivial, but subsequent work disagreed with this conclusion [4.35].

It has also been noted [4.36] that by the use of an unconventional lattice regularization scheme in which irrelevant "phantom field interactions" (such as ϕ^6, ϕ^8, etc.) are included in the action, a nontrivial ϕ^4 theory can be generated. Alternatively [4.37], a variational constraint can be applied to give an unusual "precarious" theory. Other approaches have also been tried [4.38–4.42]. It is probably fair to say that such ideas, although interesting, do not necessarily lead to a viable alternative to the canonical standard model. Time will tell.

5. Relevance for high-energy physics: Can elementary scalar particles exist?

5.1. Triviality of Higgs fields coupled to other theories

The study of triviality in pure ϕ^4 theory, as discussed in the preceding sections, is an interesting abstract problem. The question of whether scalar fields coupled to gauge theories are nontrivial is of paramount importance to our understanding of the weak interaction, for great effort is being expended in experimental searches for Higgs particles. Indeed, it can be argued that the question of the existence of Higgs particles is presently the greatest unresolved mystery in elementary particle physics, since they are at the heart of the standard model of the weak interaction.

As suggested by the discussion in the first section, if a theory is completely asymptotically free (i.e., asymptotically free in *all* coupling constants), then it is almost certainly a nontrivial theory. Simple calculations are sufficient to show that (at least in perturbation theory) completely asymptotically free theories *do* exist; these are reviewed in subsection 5.2. Unfortunately, "realistic" models theories of the weak interaction are not completely asymptotically free in general. With the exception of the so-called "eigenvalue" theories (reviewed in subsection 5.3) it seems that at least one coupling constant must be asymptotically *non*free in order to produce a realistic theory. Such a theory may still possess an ultraviolet-stable fixed point at values of the coupling constants for which perturbation theory does not well approximate its beta functions. It is for this situation that Monte Carlo renormalization group techniques were developed and applied to the standard model of the weak interaction. These methods and the results they produce are reviewed in subsection 5.4. Finally it is worth asking the question of what may be learned by considering the standard model as a low-energy effective theory. This viewpoint is discussed in subsection 5.5.

5.2. Complete asymptotic freedom in theories with Higgs particles and gauge fields

Given the likelihood of the triviality of pure ϕ^4 field theory in four dimensions, it may come as a surprise that perturbative ultraviolet fixed points *do* exist in four-dimensional theories with Higgs scalars, implying that at least some Higgs theories are nontrivial. At the present time the only fixed points which have been found in the domain of validity of perturbation theory are those for which the theory is "completely" asymptotically free, i.e. asymptotically free in all couplings. It is not known whether fixed points with couplings small enough for perturbation theory to be valid also exist, although there is no known reason why such theories should not also exist. Thus in the next sections the focus is upon the phenomenon of asymptotic freedom in theories with elementary scalars.

Asymptotic freedom in non-Abelian gauge theories was first discovered by several authors [5.1]. This remarkable phenomenon had been anticipated [5.2] in models of the strong interaction based upon the observation of Bjorken scaling in deep-inelastic electroproduction experiments at SLAC. Subsequently [5.3] it was shown that no renormalizable theory can be asymptotically free in perturbation theory without the introduction of non-Abelian gauge fields.

The property of complete asymptotic freedom in systems with Higgs particles coupled to gauge fields was then discovered [5.4] and shown to exist in a number of theories [5.5], including supersymmetric field theories [5.6]. Complete asymptotic freedom in systems which also include fermions coupled to the scalars has also been found [5.7]. This last case leads to the "eigenvalue" theories discussed in the next subsection.

In all cases, these theories were discovered by looking for systems in which all of the running coupling constants of the theory approach a fixed-point value as the renormalization point is decreased (i.e., as the "momentum transfer" increases without bound). The input required for such computations is the set of beta functions for all couplings in the theory. The evolution equations for a general theory involving scalars, spin-$\frac{1}{2}$ fermions, and vector gauge fields are given in ref. [5.5] to one-loop order, and in ref. [5.8] to two-loop order. The latter references also contain an extensive review of numerous higher-loop calculations of other quantities. A somewhat general method for locating fixed points in perturbative beta functions is given in ref. [5.9]. It has also been suggested [5.10] that if the bare quartic coupling constant of a theory vanishes, an asymptotically free theory may result.

Before examining the phenomenon of complete asymptotic freedom in specific theories it is useful to list the beta functions for the most general renormalizable theory constructed from hermitian gauge fields A^a_μ, real scalar fields ϕ_i, and spin-$\frac{1}{2}$ fields ψ_α. By assumption this theory is locally gauge invariant under some simple compact Lie group G with structure constants C^{abc}. The Lagrange density of the theory is given by

$$L = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i + \bar\psi\gamma^\mu D_\mu\psi - \bar\psi m_0\psi - \bar\psi h_i\psi\phi_i - V(\phi)\,, \tag{5.1}$$

where $V(\phi)$ is a quartic polynomial in ϕ,

$$V(\phi) = \tfrac{1}{4!}\, f_{ijkl}\phi_i\phi_j\phi_k\phi_l + \text{lower order terms}\,, \tag{5.2}$$

and where

$$F^a_{\mu\nu} \equiv \partial_\nu A^a_\mu - \partial_\mu A^a_\nu - gC^{abc}A^b_\mu A^c_\nu\,, \tag{5.3a}$$

$$(D_\mu\phi)_i \equiv \partial_\mu\phi_i + \mathrm{i}g\theta^a_{ij}\phi_j A^a_\mu\,, \tag{5.3b}$$

$$(D_\mu\psi)_\alpha \equiv \partial_\mu\psi_\alpha + \mathrm{i}gt^a_{\alpha\beta}\psi_\beta A^a_\mu\,. \tag{5.3c}$$

The beta functions of the theory describe the response of the various couplings in the theory when the renormalization point μ at which the couplings are defined is changed to $\mu\mathrm{e}^{-t}$. The equations apply in the deep Euclidean region where all dimensionful interactions (mass terms and superrenormalizable interactions) are ignored. The couplings are defined by the requirement that they are equal to the couplings which appear in the Lagrangian, defined at the symmetric points $p^2 = -\mu^2$. To one-loop order these beta functions are gauge independent [5.11] and are given by [5.5]

$$16\pi^2\,\frac{\mathrm{d}g}{\mathrm{d}t} = -[\tfrac{11}{3}S_1(G) - \tfrac{4}{3}S_3(F) - \tfrac{1}{6}S_3(S)] \equiv -\tfrac{1}{2}b_0g^3 \equiv 16\pi^2\beta_g\,, \tag{5.4a}$$

$$\begin{aligned}16\pi^2\,\frac{\mathrm{d}h_i}{\mathrm{d}t} &= 2h_mh_ih_m + \tfrac{1}{2}(h_mh_mh_i + h_ih_mh_m) + 2\,\mathrm{Tr}(h_ih_m)h_m - 3[2t^ah_it^a + S_2(F)h_i]g^2\\ &\quad + \frac{1}{288\pi^2}\, f_{ijkl}f_{jklm}h_m \equiv 16\pi^2\beta_h\,,\end{aligned} \tag{5.4b}$$

$$\begin{aligned}16\pi^2\,\frac{\mathrm{d}f_{ijkl}}{\mathrm{d}t} &= f_{ijmn}f_{mnkl} + f_{ikmn}f_{mnjl} + f_{ilmn}f_{mnjk} - 12S_2(G)g^2f_{ijkl}\\ &\quad + 3A_{ijkl}g^4 + 8\,\mathrm{Tr}(h_ih_m)\, f_{mjkl} - 12H_{ijkl} \equiv 16\pi^2\beta_f\,,\end{aligned} \tag{5.4c}$$

where

$$A_{ijkl} \equiv \alpha_{ij}^{ab}\alpha_{kl}^{ab} + \alpha_{ik}^{ab}\alpha_{jl}^{ab} + \alpha_{il}^{ab}\alpha_{jk}^{ab}\,, \quad \alpha_{ij}^{ab} \equiv \{\theta^{a}, \theta^{b}\}_{ij}\,, \tag{5.5}$$

and

$$H_{ijkl} \equiv \frac{1}{3!}\,\mathrm{Tr}[h_i h_j\{h_k, h_l\} + h_i h_k\{h_j, h_l\} + h_i h_l\{h_j, h_k\}]\,, \tag{5.6}$$

with

$$\begin{aligned} &C^{acd}C^{bcd} \equiv S_1(G)\delta^{ab}\,, \quad (t^a t^a)_{ij} \equiv S_2(F)\delta_{ij}\,, \quad (\theta^a\theta^a)_{ij} \equiv S_2(S)\delta_{ij}\,, \\ &\mathrm{Tr}(t^a t^b) \equiv S_3(F)\delta^{ab}\,, \quad \mathrm{Tr}(\theta^a\theta^b) \equiv S_3(S)\delta^{ab}\,. \end{aligned} \tag{5.7}$$

Equations (5.7) define constants S_1, S_2, and S_3 which depend only upon the group (G) and the respective representations of the fermions (F) and scalars (S). The fact that the tensor f_{ijkl} is totally symmetric has also been used. Shifting the scalar fields to have vanishing vacuum expectation values and give masses to the gauge particles and fermions only affects superrenormalizable terms in the Lagrangian. The renormalization group equations are thus unaffected by the presence of spontaneous symmetry breaking. (More generally, it has been shown [5.12] that if a theory exhibits both complete asymptotic freedom and spontaneous symmetry breaking, the Green's functions in the deep Euclidean region are the same as those of the symmetric theory.)

Some intuition for the nature of the solutions of these equations can be developed by the exploration of specific examples. Consider (following ref. [5.5]) the case of an O(N) gauge theory interesting only with a single fundamental (i.e. vector) representation scalar. The most general O(N)-invariant quartic coupling can be written [cf. eq. (5.2)]

$$V(\phi) = \tfrac{1}{8}\lambda(\phi\cdot\phi)^2\,. \tag{5.8}$$

The renormalization group equations for λ and g^2 are

$$16\pi^2\,\frac{\mathrm{d}\lambda}{\mathrm{d}t} = (N+8)\lambda^2 - 3(N-1)g^2\lambda + \tfrac{3}{4}(N-1)g^4\,, \tag{5.9a}$$

$$16\pi^2\,\frac{\mathrm{d}(g^2)}{\mathrm{d}t} = -b_0(g^2)^2\,, \tag{5.9b}$$

with the initial conditions $\lambda(t=0) \equiv \lambda_{\mathrm{R}}$, $g^2(t=0) \equiv g_{\mathrm{R}}^2$. It is natural to define new variables

$$\mathrm{d}s \equiv \frac{N+8}{16\pi^2}\,g^2\,\mathrm{d}t\,, \qquad \frac{\lambda}{g^2} \equiv \zeta\,, \tag{5.10}$$

so that (imposing the condition that s vanishes when t does)

$$\frac{e^{\varepsilon s}-1}{\varepsilon} = \left(\frac{g_{\mathrm{R}}^2(N+8)}{16\pi^2}\right)t\,, \tag{5.11a}$$

$$g^2 = g_R^2\, e^{-\varepsilon s}\,, \tag{5.11b}$$

$$d\zeta/ds = (\zeta - \zeta_+)(\zeta - \zeta_-)\,, \tag{5.11c}$$

with

$$\varepsilon \equiv b_0/(N+8)\,. \tag{5.11d}$$

Solutions of eqs. (5.11) exist for which both λ and g^2 are asymptotically free, i.e., where λ and g^2 approach zero as t (and therefore s) increases without bound. In order to produce this behavior b_0 must be positive and the roots $\zeta_\pm$ must be real. It is thus most advantageous to choose b_0 as small as possible (while remaining positive). This can be accomplished by adding fermions to the theory. The results do not change significantly if b_0 is set to zero. The requirement that the roots $\zeta_\pm$ be real then forces the discriminant of the quadratic equation (5.11d) to be positive,

$$3(N-1)(2N-11) > 0\,. \tag{5.12a}$$

Assume that eq. (5.12a) holds, and choose ζ_+ to be the larger of the two roots $\zeta_\pm$. Then as t increases without bound, eq. (5.11c) implies that

$$\begin{aligned} &\zeta \to \zeta_-\,, \quad \text{if } \zeta_R < \zeta_+\,, \\ &\zeta = \zeta_+\,, \quad \text{if } \zeta_R = \zeta_+\,. \end{aligned} \tag{5.12b}$$

[Note that corrections to eq. (5.12b) are of higher order in g^2, and thus presumably vanish in the limit of large t.] The fact that $\zeta = \lambda/g^2$ approaches a fixed ratio in this limit implies that λ and g^2 are both asymptotically free if $\zeta_R \le \zeta_+$ and $b_0 > 0$; and thus the theory is presumably nontrivial.

It is interesting to note that if the "spontaneously" generated mass of the gauge bosons is denoted by m_W and the mass of the scalar particle is denoted by m_H, eqs. (5.12b) imply the bound [5.13, 5.14]

$$(m_H/m_W)^2 \le \zeta_+ \tag{5.13}$$

(again, to lowest order in the loop expansion). Equation (5.13) assumes that no further fixed point exists. Upper bounds on the masses of Higgs particles in more general theories may arise from the requirement of a nontrivial Higgs sector and are discussed in subsection 5.5. The analysis leading to eq. (5.13) suggests that gauge fields can rescue a pure ϕ^4 theory from triviality, and that the requirement that the full theory be nontrivial may generate experimental predictions.

What is the group structure of models with complete asymptotic freedom? From eq. (5.12a), it follows that N must be at least six for complete perturbative asymptotic freedom to occur. Yet in an O(N) gauge theory a single fundamental representation scalar breaks the gauge symmetry only from O(N) to O($N-1$), so that only in an O(2) gauge model is the symmetry broken completely. For this example of an O(N) gauge theory, the smallest group which can be completely asymptotically free is O(6). The gauge group is spontaneously broken to O(5) [5.5] and there are still ten massless gauge particles corresponding to the generation of the remaining *non-Abelian* O(5) group.

This observation seems to be part of a general pattern—if the theory is completely asymptotically free, there are a large number of *massless* gauge bosons present, corresponding to a large non-Abelian subgroup which remains unbroken. Exceptions to this rule include the "eigenvalue" theories discussed in the next subsection, in which every renormalized coupling in theory is determined, and in which complete asymptotic freedom and complete spontaneous symmetry breaking can coexist.

Further examples are useful in order to make this behavior explicit. A more general case of the above theory occurs when an O(N) gauge theory is coupled to M different fundamental representation scalars. The minimum critical value for complete asymptotic freedom in an O(N) gauge model, $N_c(M)$, is given in Table 5.1 as a function of M. Note that in each case the O(N) gauge symmetry is broken to O($N-M$). For large N, complete asymptotic freedom occurs for $M < N-1 \approx N$. Thus again it is seen that these two phenomena are generally incompatible with one another. Nor does the pattern seem to change if [5.5] second-rank tensor representations of Higgs fields are used (table 5.2), or if [5.11, 5.5] the gauge group is SU(N) (tables 5.3, 5.4). In all cases the Higgs mechanism fails to remove the infrared singularities, and in no case is the gauge group broken down to an ultraviolet-stable Abelian symmetry. Moreover, the situation becomes progressively worse for scalars in higher group representations, since [5.5] the contributions $S_1(G)$ to the running gauge coupling (5.4a) have the right sign and are proportional to N for both O(N) and SU(N), while the scalar contributions $S_3(S)$ have the wrong sign and are proportional to N^{k-1} for scalars belonging to kth rank tensor representations. Thus for large enough N, a tensor higher than the second rank ultimately must destabilize the gauge coupling and prevent complete asymptotic freedom.

In order to form a complete model of the weak interaction, it is necessary to consider the effects of Yukawa couplings. These couplings are generally driven to zero at least as fast as the gauge couplings at a fixed point where complete asymptotic freedom occurs. Thus the hf^2 terms in the evolution equation (5.4b) can be dropped. This behavior can be seen [5.5] from the evolution equations in a case where only a single Yukawa coupling is present,

$$16\pi^2 \frac{dh}{dt} = Ah^3 - Bg^2h\,, \tag{5.14}$$

where A and B are positive constants which depend upon group-theoretic factors. If a new variable $\tilde{h} \equiv h/g$ is defined, then eqs. (5.14) and (5.4a) imply that

$$\frac{16\pi^2}{g^2}\frac{d\tilde{h}}{dt} = \tilde{h}[\tilde{h}^2 - (B - b_0)]\,. \tag{5.15}$$

Table 5.1
Minimum value N_c for complete asymptotic freedom in an O(N) gauge theory with M fundamental representation Higgs particles

M	N_c	Symmetry breaking
1	6	O(N)→O($N-1$)
2	7	O(N)→O($N-2$)
3	7	O(N)→O($N-3$)
4	8	O(N)→O($N-4$)
5	9	O(N)→O($N-5$)
6	10	O(N)→O($N-6$)
7	11	O(N)→O($N-7$)

Table 5.2
Minimum value N_c for complete asymptotic freedom in an O(N) gauge theory coupled to tensor representation Higgs particles

Representation	N_c
Antisymmetric second-rank tensor	8
Antisymmetric second-rank tensor + fundamental scalar	9
Symmetric second-rank tensor	14

Table 5.3
Minimum value N_c for complete asymptotic freedom in an SU(N) gauge theory with M fundamental representation Higgs particles

M	N_c	Symmetry breaking
1	3	$SU(N) \to SU(N-1)$
2	4	$SU(N) \to SU(N-2)$
3	5	$SU(N) \to SU(N-3)$
4	6	$SU(N) \to SU(N-4)$
5	7	$SU(N) \to SU(N-5)$
6	8	$SU(N) \to SU(N-6)$
7	10	$SU(N) \to SU(N-7)$
8	11	$SU(N) \to SU(N-8)$

Table 5.4
Minimum value N_c for complete asymptotic freedom in an SU(N) gauge theory coupled to tensor representation Higgs particles

Representation	N_c
Adjoint	6
Adjoint + fundamental	7
Symmetric tensor	9

The fixed point at $\tilde{h}$ equal to zero is ultraviolet stable, so $h(t)$ vanishes faster than $g(t)$ for large t, provided that

$$B > b_0 , \qquad h_R^2 < (B - b_0) g_R^2 . \tag{5.16}$$

The conditions eqs. (5.16) are easily enough satisfied, and imply that the Yukawa couplings can be neglected in the consideration of theories which are completely asymptotically free, provided that the Yukawa couplings are not too large.

A second possibility is that

$$h_R^2 = (B - b_0) g_R^2 , \tag{5.17a}$$

in which case

$$\mathrm{d}\tilde{h}/\mathrm{d}t = 0 . \tag{5.17b}$$

Specific solutions [of the general form eqs. (5.17)] do in fact exist in coupled systems involving gauge, quartic, and Yukawa couplings. These "eigenvalue" solutions (discussed in the next subsection) require that specific relations exist between all renormalized couplings in the theory.

5.3. Eigenvalue conditions for asymptotic freedom in models of the weak interaction

Probably the simplest example of a theory whose coupling constants have evolution equations possessing [5.7] an "eigenvalue" solution for asymptotic freedom is the Georgi–Glashow model of the weak interaction [5.15]. In this model the electron and its neutrino are combined with two new heavy particles: a charged lepton, E^+, and an uncharged lepton, X^0, to make left- and right-handed SO(3) triplets. Similarly, heavy leptons M^+ and Y^0 are added to make muon triplets. This model possesses only two charged weak gauge fields (denoted $W_\mu^\pm$) and the electromagnetic gauge field A_μ. Since it therefore fails to account for neutral weak leptonic currents, it is not a viable candidate for a realistic model of the weak interaction. Nevertheless, it does provide an example of the idea of eigenvalue conditions for asymptotic freedom.

Define the quartic coupling constant λ by an interaction Lagrange density for the scalar field,

$$L(\phi) = \frac{-\lambda}{4!}\,\phi^4 . \tag{5.18}$$

Denote the vacuum expectation value of the scalar field by v, and the ν_e–X_0 mixing angle by θ_e. Similarly, define θ_μ to be the corresponding muonic mixing angle. Then four Yukawa couplings (h_1, h_2, H_1, H_2) can be defined in terms of particle masses by

$$m_{E^+} - m_{e^-} = 2vh_1 , \tag{5.19a}$$

$$m_{X^0} = \frac{h_2 v}{\sin\theta_e} , \tag{5.19b}$$

$$m_{M^+} - m_{\mu^-} = 2vH_1 , \tag{5.19c}$$

$$m_{Y^0} = \frac{H_2 v}{\sin\theta_\mu} . \tag{5.19d}$$

The lowest-order equations for the evolution of the couplings are then given by [5.7]

$$16\pi^2 \frac{dg^2}{dt} = -b_0 g^4 = -[\tfrac{44}{3} - \tfrac{16}{3}(2) - \tfrac{2}{3}(1)]g^4 = -\tfrac{10}{3}g^4 , \tag{5.20a}$$

$$16\pi^2 \frac{dh_1^2}{dt} = 16h_1^4 + 4h_1^2(2H_1^2 + H_2^2) + h_1^2h_2^2 - 24g^2h_1^2 , \tag{5.20b}$$

$$16\pi^2 \frac{dh_2^2}{dt} = 12h_2^4 + 4h_2^2(2H_1^2 + H_2^2) + 2h_1^2h_2^2 - 12g^2h_2^2 , \tag{5.20c}$$

$$\frac{d\theta_e}{dt} = \frac{d\theta_\mu}{dt} = 0 , \tag{5.20d}$$

$$16\pi^2 \frac{d\lambda}{dt} = \tfrac{11}{3}\lambda^2 + (16h_1^2 + 8h_2^2 + 16H_1^2 + 8H_2^2 - 24g^2)\lambda + 72g^4 - 96(h_1^4 + H_1^4) - 48(h_2^4 + H_2^4) . \tag{5.20e}$$

The equations for H_1^2 and H_2^2 are obtained from eqs. (5.20b) and (5.20c) by μ–e symmetry. Asymptotically free eigenvalue solutions exist for which

$$\begin{aligned} &h_1^2 = k_1 g^2 , \qquad H_1^2 = K_1 g^2 , \quad \lambda = \Lambda g^2 \\ &h_2^2 = k_2 g^2 , \qquad H_2^2 = K_2 g^2 , \end{aligned} \tag{5.21}$$

and k_1, k_2, K_1, K_2, and Λ are numerical constants. One solution is [5.7]

$$k_1 = K_1 = (324 - 11b_0)/334 , \qquad k_2 = K_2 = (34 - 7b_0)/167 , \tag{5.22a,b}$$

while the constant Λ is given by the positive root of the quadratic

$$\tfrac{11}{3}\Lambda^2 + (32K_1 + 16K_2 - 24 + b_0)\Lambda - (192K_1^2 + 96K_2 - 72) = 0 . \tag{5.22c}$$

Equation (5.22c) possesses such a positive root provided that $192K_1^2+96K_2>72$, which is amply satisfied in a purely leptonic theory. Corrections to eqs. (5.22) can be expressed as a power series in g^2 [5.7],

$$h^2 = K^{(0)}g^2 + K^{(1)}g^4 + K^{(2)}g^6 + \cdots . \tag{5.23}$$

The solution (5.22) of the evolution equations has the property that all of the renormalized parameters of the theory are determined from two constants, which can be adjusted to fit experimental data. In the Georgi–Glashow model, μ–e symmetry is strictly observed. It is therefore appropriate to use as input an "average" mass for $m_e = m_\mu = 0.053$ GeV. If the gauge coupling $g_R \equiv g(t=0)$ is identified with the electromagnetic coupling constant, the "physical" predictions [5.7]

$$\begin{aligned} &m_{X^0} = m_{Y^0} = 3.34\ \mathrm{GeV}/c^2\,, \qquad m_{W^+} = 3.58\ \mathrm{GeV}/c^2\,, \qquad \theta \simeq 0.0669(1-0.001/0.97)\,,\\ &m_{E^+} = m_{M^+} = 6.64\ \mathrm{GeV}/c^2\,, \qquad \lambda_R \simeq 3.52 g_R^2\,, \end{aligned} \tag{5.24}$$

are generated. Note that heavy leptons with masses of the order of magnitude of the W are predicted. The existence of heavy leptons is a typical feature of eigenvalue theories.

It is important to point out that other eigenvalue solutions to the evolution equations also exist. These solutions have zero eigenvalues, and are given by [5.7]

$$\text{(I)} \quad k_1 = K_1 = 1 - b_0/24\,, \qquad k_2 = K_2 = 0\,, \tag{5.25a}$$

$$\begin{aligned} &\text{(II)} \quad k_1 = \tfrac{111}{118} - \tfrac{23}{944}b_0\,, \qquad K_1 = (240-11b_0)/236\,, \qquad k_2 = 0\,, \qquad K_2 = (24-7b_0)/118\,,\\ &\text{(III)} \quad k_1 = 0\,, \qquad K_1 = (2268-77b_0)/1491\,, \qquad k_2 = (4032-413b_0)/1491\,, \end{aligned} \tag{5.25b}$$

$$K_2 = (180-14b_0)/213\,. \tag{5.25c}$$

[Two other solutions exist, which are μ↔e mirror images of (II) and (III).] These solutions predict unphysical behavior [e.g. (III) implies that the heavy lepton E^+ is degenerate in mass with the electron]. Moreover, solutions exist for which h_1^2/g^2 approaches a constant but h_2^2/g^2 vanishes like a power of g^2 when g^2 approaches zero [5.7]. However, no ultraviolet-*stable* solutions of the form (5.11) exist—in all cases perturbative asymptotic freedom occurs only when certain relations between the renormalized couplings of the theory hold. The "eigenvalue" fixed points are thus ultraviolet *un*stable—small perturbations in the parameters k_1, k_2, K_1, and K_2 move the theory away from the fixed point. It is not known whether other (nonasymptotically free) fixed points exist. If they do not, then this theory provides an example of a case in which the requirement of a nontrivial theory allows one to predict the masses of various particles.

The Georgi–Glashow SO(3) model is not the only theory for which such eigenvalue conditions exist. Eigenvalue relations for complete asymptotic freedom have also been derived in theories whose gauge symmetry group is SU(2) [5.16], SU(3) [5.17], SU(2) × SU(2) × SU(4) [5.18], SU(5) [5.19], SO(N) [5.20], and E_6 [5.21]. Thus it can be seen that the idea of eigenvalue solutions at a fixed point is quite general.

The discussion in the above two subsections makes it clear that gauge fields may well turn a trivial pure ϕ^4 field theory into a nontrivial coupled system. Ultraviolet-stable fixed points [like those in eq. (5.11)], and ultraviolet-unstable (eigenvalue) fixed points can appear in coupled systems, even though none existed in the pure scalar theory. Since these fixed points correspond to asymptotically free theories, the dynamics of the system can be analysed perturbatively.

In order for a field theory to be asymptotically free in perturbation theory, non-Abelian gauge fields must be present [5.3]. This statement is an oft quoted result, but implicitly assumes that the quartic self-coupling f_{ijkl} in eq. (5.4c) is a positive-definite tensor such that the energy spectrum is bounded from below. This requirement is less obvious in a system where other fields are coupled to the scalars [5.22]. Arguments based upon the renormalization group, however, do suggest that such a constraint is needed [5.23].

As discussed above, complete spontaneous symmetry breaking (where the remaining unbroken subgroup is Abelian) is not in general consistent with complete asymptotic freedom. Realistic models of the weak interaction therefore generally require at least one coupling constant to be *not* asymptotically free. In order to determine whether or not such a theory is nontrivial, a computational framework must be developed which is independent of perturbation theory. One viable scheme is provided by the application of the renormalization group to lattice gauge theories, and is discussed in detail in the next subsection.

5.4. *Nonperturbative renormalization group analysis of the standard model: Is the Higgs mass predictable?*

In the last subsection it has been pointed out that the study of triviality in theories like the standard model (where at least one coupling is not asymptotically free) requires nonperturbative methods. Lattice gauge theory [5.24–5.26] provides a framework for such an analysis, since the ultraviolet cutoff (the lattice spacing) is defined independently of perturbation theory. References [5.24–5.32] review the tenets of the discipline, while ref. [5.33] presents an introduction to the subject for nonspecialists.

To these lattice gauge theories, it is possible to apply the Monte Carlo renormalization group ([3.23, 3.2], also reviewed in section 3). The basic idea is that in a lattice gauge theory a continuum limit requires a divergent correlation length (i.e. a vanishing lattice spacing) and thus must occur (if at all) at a phase transition of second or higher order. Any continuum limit therefore lies on a critical surface. Thus an analysis of these limits is a study in critical phenomena, for which this renormalization group was developed (see section 3).

Recall that in this renormalization group the original system is divided up into "blocks" or groups of variables. This new "block" system is governed by an action defined by a set of block coupling constants $\{K'\}$, just as the original action is defined by a set of couplings $\{K\}$. As the system is blocked repeatedly, a set of "flows" $\{K\} \to \{K'\} \to \{K''\} \to \cdots$ is mapped out for all $\{K\}$. If the starting $\{K\}$ are on a critical surface, the block couplings $\{K'\}$ will also be on a critical surface, since under the blocking transformation (by a linear scale factor b)

$$\{K\} \to R_b[\{K\}] = \{K'\} \tag{5.26a}$$

and the dimensionless correlation length ξ scales according to

$$b\xi\{K'\} = \xi\{K\} . \tag{5.26b}$$

Thus if ξ is infinite, it will remain so under a blocking transformation.

As the blocking transformation is applied repeatedly, the degrees of freedom of the system are progressively integrated out, and the coupling constants typically flow towards fixed points. At such fixed points, $\{K'\} = \{K\} = \{K^*\}$, and so by eq. (5.26b) the dimensionless correlation length ξ is either infinite or zero. From the flows in the neighborhood of a critical fixed point (specifically from the eigenvalues of the derivative matrix $D_{ab} \equiv \partial K'_a/\partial K_b$) the anomalous dimensions of the field theories defined on the critical surface associated with the fixed point can be determined. A trivial theory should have vanishing anomalous dimensions.

A second important point is that (as discussed in section 3) the number of *relevant* parameters for the continuum theories associated with a given fixed point is equal to the number of its independent renormalized couplings. Here, the word "relevant" refers to the number of directions in which the renormalization group trajectories flow away from the fixed point. Thus if the number of relevant couplings for a fixed point is less than the number of renormalized coupling constants (i.e., if at least one direction is irrelevant) then one or more of these renormalized couplings in the continuum theory can be calculated from the others.

This quantum "parameter reduction" [5.13] occurs in the pure ϕ^4 case at the gaussian fixed point, where the quartic coupling constant is irrelevant and thus calculable (in this case it is zero). Another example of the phenomenon of a "measure-zero" fixed point occurs in the eigenvalue theories discussed in the last section. Moreover, if a nontrivial continuum limit of the lattice standard model exists and if (as is predicted by universality) the quartic coupling is irrelevant at this fixed point, *the standard model Higgs mass can be predicted from the other parameters of the theory.*

5.4.1. The lattice standard model

An important consideration in the analysis of the nonperturbative renormalization group structure of the weak interaction is the definition of the lattice standard model. The version discussed here [5.34, 5.35] is the simplest formulation in the tradition of Wilson [5.24]. Further analyses of the full standard model using other lattice formulations should clearly be performed (see refs. [5.36] in this regard).

In the lattice standard model considered here, the fermionic sector of the model is neglected. This simplification may be impossible to avoid, for the inclusion of chiral fermions in a lattice theory is an extraordinarily difficult problem [5.36, 5.29]. If only light fermions are considered, however, the associated Yukawa couplings are quite small (e.g., $\sim 10^{-5}$ for the electron) and are arguably negligible. Fermions whose masses are an appreciable fraction of the ~200 GeV mass scale of the weak interaction may require further consideration.

The bosonic sector of the lattice standard model can be defined by the partition function (the vacuum-to-vacuum transition amplitude) [5.34],

$$ {}_{\infty}\langle 0|0\rangle_{-\infty} \equiv Z \equiv \int \left[\prod \mathrm{d}U\right]\left[\prod \mathrm{d}V\right]\left[\prod \mathrm{d}\Phi^{+}\right]\left[\prod \mathrm{d}\Phi\right] \mathrm{e}^{-S_{\mathrm{var}}} , \tag{5.27a} $$

where

$$ S_{\mathrm{var}} = \beta_1 S_1 + \beta_2 S_2 + S_{\mathrm{V}} + S_{\mathrm{H}} , \tag{5.27b} $$

$$ S_1(U) = \sum_{\mathrm{plaq}} (1 - \mathrm{Re}\, U_{\mathrm{plaq}}) , \tag{5.27c} $$

$$ S_2(V) = \sum_{\mathrm{plaq}} (1 - \tfrac{1}{2}\, \mathrm{Tr}\, V_{\mathrm{plaq}}) , \tag{5.27d} $$

$$S_{\mathrm{V}}(U, V, \Phi) = -2 \sum_{\mu,\nu} \mathrm{Re}(\Phi_n^+ U_{n,\mu} V_{n,\mu} \Phi_{n+\mu}) , \tag{5.27e}$$

$$S_{\mathrm{H}}(\Phi) = \lambda \sum_{n} (\Phi_n^+ \Phi_n - \kappa)^2 . \tag{5.27f}$$

Here, U_{plaq} and V_{plaq} denote path-ordered products of links $U_{n,\mu}$ and $V_{n,\mu}$ defined in the fundamental representations of the groups $U(1)$ and $SU(2)$, respectively. The links connect lattice site n to lattice site $n + \mu$ (μ is a direction, n a lattice coordinate). Specifically,

$$U_{\mathrm{plaq}} = U_{n,\mu} U_{n+\mu,\nu} U^+_{n+\nu,\mu} U^+_{n,\mu} , \qquad V_{\mathrm{plaq}} = V_{n,\mu} V_{n+\mu,\nu} V^+_{n+\nu,\mu} V^+_{n,\mu} , \tag{5.28}$$

where μ and ν represent orthogonal directions. The sum refers to a sum over all such elementary plaquettes in the lattice.

The link $U_{n,\mu}$ can be written as a pure phase,

$$U_{n,\mu} = \exp(\mathrm{i}\theta_{n,\mu}) . \tag{5.29a}$$

The Haar measure for this compact Lie group is

$$\int \mathrm{d}U_{n,\mu} \equiv \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta_{n,\mu}}{2\pi} . \tag{5.29b}$$

Similarly, the SU(2) links $\{V\}$ can be parametrized in terms of the 2×2 Pauli matrices σ and the identity matrix $\mathbf{1}$

$$V = a_0 \mathbf{1} + \boldsymbol{a} \cdot \boldsymbol{\sigma} , \tag{5.30}$$

where a_μ is a four-component object satisfying the relation

$$a^2 \equiv a_\mu a^\mu \equiv a_0^2 + \boldsymbol{a}^2 = 1 , \tag{5.31a}$$

so that all V have unit determinant. The Haar measure for the group can then be written [5.37]

$$\int \mathrm{d}V \equiv \int \frac{\mathrm{d}^4 a}{2\pi^2} \, \delta(a^2 - 1) . \tag{5.31b}$$

The scalar field Φ_n is a complex ($I_w = \frac{1}{2}$, $Y = 1$) Higgs field associated with each site n,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{5.32}$$

The model defined by eqs. (5.27)–(5.32) has [5.34] the correct formal continuum limit of the standard model, provided that the definitions

$$\beta_1 = 1/g_1^2 , \qquad \beta_2 = 4/g_2^2 \tag{5.33}$$

are made, with g_1 and g_2 the bare lattice gauge coupling constants for the U(1) and SU(2) gauge groups, respectively. The coefficient of the $\phi^+\phi$ term in the Lagrange density is $-2\lambda\kappa$ (in units where the lattice spacing is unity), with λ the bare lattice quartic coupling.

It is useful to introduce the simplification of a fixed-length lattice scalar field. This reduction is equivalent to taking the limit of large λ in eqs. (5.27). Since the resulting (fixed-length) model is in the same universality class as the variable-magnitude model, this simplification should not affect the critical behavior of the model. Following the rescaling $\Phi \to \sqrt{\kappa}\,\Phi$, the action is given by

$$S = \beta_1 S_1 + \beta_2 S_2 + \kappa S_{\mathrm{L}}\,, \tag{5.34a}$$

where

$$S_{\mathrm{L}} = -2 \sum_{n,\mu} \mathrm{Re}(\Phi_n^+ U_{n,\mu} V_{n,\mu} \Phi_{n,\mu})\,, \tag{5.34b}$$

subject to the constraint

$$\Phi_n^+ \cdot \Phi_n = 1 \quad \text{for all } n\,. \tag{5.34c}$$

This theory possesses [5.34] three phases:

(1) a *confined* phase, in which the free energy required to separate a pair of test charges increases without bound as the separation between them grows;

(2) an *electrodynamics* or "Coulomb" phase, where interparticle forces between test charges obey Coulomb's law; and

(3) a "*Higgs*" phase, where spontaneous symmetry breaking occurs.

5.4.2. The SU(2)–Higgs limit model

Before discussing the structure of the renormalization group flows of the full standard model, it is worthwhile to exhibit the formalism in the simpler case of the SU(2)–Higgs model. This model is the limit of the fixed-length model described by the action eqs. (5.34) in the limit where β_1 increases without bound (i.e., where the U(1) gauge coupling g_1 approaches zero). It is a theory with only elementary scalar fields and SU(2) gauge fields, and is described by an action

$$S_{\mathrm{SU}(2)} = \beta_2 S_2 + \kappa \tilde{S}_{\mathrm{L}}\,, \tag{5.35a}$$

where

$$\tilde{S}_{\mathrm{L}} \equiv -2 \sum_{n,\mu} \mathrm{Re}(\Phi_n^+ V_{n,\mu} \Phi_{n+\mu})\,, \tag{5.35b}$$

with the other conventions as established previously. This and related models have been studied extensively by several authors [5.38–5.43].

The most relevant feature of a lattice gauge theory is its phase structure, since a divergent correlation length (and thus a phase transition of second or higher order) is required for a continuum limit. Some interesting properties of the phase structure of this theory are known. First, in the large-β_2 limit of vanishing SU(2) gauge coupling, the theory becomes a nonlinear sigma model with O(4) symmetry. This model has a phase transition at a critical coupling κ_c.

The exact value of this critical coupling κ_c is not known, but a rigorous lower bound is provided [5.44] by the mean-field theory estimate,

$$2\kappa_c = \beta^c_{XY} \geq \tfrac{1}{2}\,, \tag{5.36}$$

while a rigorous upper bound is given by [5.45, 5.34]

$$2\kappa_c = \beta^c_{XY} \leq 0.622\,. \tag{5.37}$$

Additionally, an expansion in powers of the inverse coordination number q^{-1} (for a hypercubic lattice $q = 8$) yields [5.46, 5.34]

$$2\kappa_c = \beta^c_{XY} \cong 0.6055 + O(q^{-6})\,, \tag{5.38}$$

where the q^{-5} term makes a fractional contribution of 5×10^{-3}. In eqs. (5.36)–(5.38) the connection with the standard notation β_{XY} associated with the XY model has been made explicit.

The phase transition itself is of second order, with $\nu = \frac{1}{2}$. Thus the anomalous dimension γ_{Φ^2} of the operator Φ^2 is zero (see section 3),

$$\gamma_{\Phi^2} = 2 - \nu^{-1} = 0\,. \tag{5.39}$$

The specific heat $C(\tau)$ diverges weakly as [5.47]

$$C(\tau) \sim \log(\log(\tau))\,, \qquad \tau \equiv |(\kappa - \kappa_c)/\kappa_c|\,. \tag{5.40}$$

in the limit of small $|\tau|$.

Evidence from numerical and other approximate methods suggests that there is a strong phase transition extending from this point (labelled "G" in fig. 5.1) into the interior of the phase diagram [5.38–5.43]. This line of phase transitions ends at a critical point (labelled "M" in fig. 5.1). The order of this phase transition remains an unknown and controversial issue, although it is probably fair to say that it is a first-order transition or possibly a very strong second-order transition.

Monte Carlo renormalization group techniques must be applied to this model with some care in order to generate a consistent flow diagram. The point is that most traditional blocking procedures (see, e.g., ref. [5.48]) for gauge fields involve each original link in the definition of several block links.

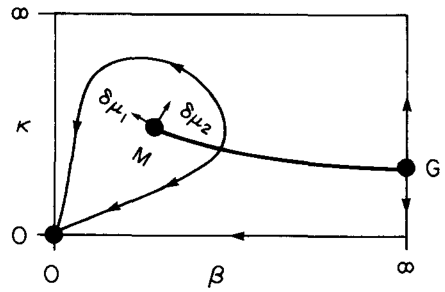

Fig. 5.1. Flow diagram of the SU(2)-Higgs model.

Specifically, the most obvious blocking method is to make use of the fact that the sum of SU(2) elements V differs from another SU(2) element by at most an overall normalization factor. This serendipity suggests a definition of the block links of the form

$$V'_{n,\mu} = \frac{V_{\text{sum}}}{\det(V_{\text{sum}})} , \tag{5.41}$$

where

$$V_{\text{sum}} = \sum_{\text{path}} V_{\text{path}} , \tag{5.42}$$

and V_{path} refers to an ordered product of V's between the two block sites connected by $V'_{n,\mu}$. Thus the block link has the appropriate transformation properties under gauge rotations.

Unfortunately when several paths between two block points are thus averaged, the result includes each original link in several block links. Spurious interactions are generated between block links, which lead to an inconsistent flow diagram [5.35, 5.39, 5.40]. A solution to this problem is to block with sets of paths which are chosen so that each link is included in only one block link [5.49]. Expectation values in the blocked system can also be defined as averages over ensembles of different blocking schemes [5.40]. The scalar fields can be blocked as well, by taking the suitably normalized average of the original scalar fields in the block, appropriately parallel transported [5.35, 5.39].

After this blocking is complete, a new set of block variables $\{\Phi'\}$ and $\{V'\}$ is defined for a given set of $\{\Phi\}$ and $\{V\}$. If the original configurations are generated with statistical weight

$$\mathrm{e}^{-S\{V,\Phi\}}\, \mathrm{D}V\, \mathrm{D}\Phi\, \mathrm{D}\Phi^{+} , \tag{5.43}$$

where

$$\mathrm{e}^{-S'\{V',\Phi'\}} = \int \mathrm{D}V\, \mathrm{D}\Phi\, \mathrm{D}\Phi^{+}\, P\, \mathrm{e}^{-S\{V,\Phi\}}, \tag{5.44}$$

and $P = P[\{V', \Phi'\}; \{V, \Phi\}]$ is the projection operator which defines the blocking transformation. The requirement that the partition function [cf. eq. (5.27a)] is preserved by the blocking imposes the constraint

$$\int \mathrm{D}V'\, \mathrm{D}\Phi'\, \mathrm{D}\Phi^{+\prime}\, P = 1 . \tag{5.45}$$

The projection operator is otherwise arbitrary.

The problem at this point is to calculate the block action $S'\{V', \Phi'\}$ given a set of configurations which are generated with the weight (5.43). Various methods have been proposed for this general problem [5.50, 5.51, 5.39, 5.35, 5.40]. One technique that seems particularly well-suited to the present problem makes use of a set of Schwinger–Dyson identities in the block system [5.35, 5.51].

The idea of this method for the extraction of the block action is to use the principle of gauge invariance to generate consistency equations involving various block coupling constants. Consider the effect of infinitesimal SU(2) gauge rotations R_ε on the block scalar and link variables,

$$R_\varepsilon \Phi'_n = (1 + \mathrm{i}\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma})\Phi'_n \equiv \Phi'_n + \boldsymbol{\varepsilon} \cdot \boldsymbol{\Delta}_n \Phi'_n \,, \tag{5.46a}$$

$$R_\varepsilon V'_{n,\mu} = (1 + \mathrm{i}\boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma})V'_{n,\mu} \equiv V'_{n,\mu} + \boldsymbol{\varepsilon} \cdot \boldsymbol{\Delta}_{n,\mu} V'_{n,\mu} \,, \tag{5.46b}$$

where $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_{n,\mu}$ are differential operators. Because the measure is gauge invariant, the following identites hold in the block system:

$$\int \mathrm{D}V'\, \mathrm{D}\Phi^{+\prime}\, \mathrm{D}\Phi'\, \boldsymbol{\Delta}_n \cdot [(\boldsymbol{\Delta}_n A_m)\, \mathrm{e}^{-S'_{\mathrm{SU}(2)}}] = 0 \,, \tag{5.47a}$$

$$\int \mathrm{D}V'\, \mathrm{D}\Phi^{+\prime}\, \mathrm{D}\Phi'\, \boldsymbol{\Delta}_{n,\mu} \cdot [(\boldsymbol{\Delta}_{n,\mu} B_m)\, \mathrm{e}^{-S'_{\mathrm{SU}(2)}}] = 0 \,, \tag{5.47b}$$

where the A_m and B_m are gauge-invariant functionals of the $\{V'\}$ and $\{\Phi'\}$. The relations eqs. (5.47) are *exact* in the block system, and yield an infinite set of identities. In particular, the A_m and B_m can be chosen as an infinite orthogonal basis set spanning the space of all possible block actions, so that

$$S'\{V, \Phi\} = \sum_m (c_m A_m + d_m B_m) \,, \tag{5.48}$$

with the c_m and d_m denoting appropriate coupling constants. Then eqs. (5.47) can be used to express the c_m and d_m in terms of expectation values over the $\{V', \Phi'\}$ ensemble. Thus the complete set of block couplings $\{c, d\}$ which specifies the block action can be determined.

This task would be vastly simplified if the block action were of the same form as the original action,

$$S'_{\mathrm{SU}(2)}\{V', \Phi'\} = \beta'_2 B_0 + \kappa' A_0 \,, \quad A_0 = \tilde{S}'_{\mathrm{L}}\{V', \Phi'\} \,, \quad B_0 = S'_2\{V'\} \,. \tag{5.49}$$

If eqs. (5.49) are substituted into eqs. (5.47), the result is a two-by-two set of linear equations, from which β'_2 and κ' can be calculated in terms of expectation values in the block system. Essential to this calculation is the assumption that the block action is precisely of the form eqs. (5.49). An alternative point of view is to assume that eqs. (5.49) are a good approximation to the block action, and then simply define *effective* couplings β'_2 and κ' by the combination of eqs. (5.47) with eqs. (5.49).

This "maximal" truncation of the blocked action to two couplings may be a good approximation if the blocking scheme is sufficiently comprehensive. However, it must be pointed out that although such approximations can be systematically improved by the inclusion of more terms in the block action, no general method for the analysis of errors generated by this truncation is known. Clearly much more work needs to be done on the optimization of the blocking transformation [5.52].

Nevertheless, such truncated calculations do yield a simple conceptual framework for understanding the nonperturbative structure of quantum field theory. This can be illustrated by considering the results of such calculations for the SU(2)–Higgs model. Blocking transformations with scale factor $b = \sqrt{2}$ [5.40], $b = 2$ [5.39], and $b = 3$ [5.35] have been applied to this model. In all cases, the flow diagram was of the qualitative form shown in fig. 5.1.

In these simulations, a fixed point G appears at infinite β_2 and $\kappa \approx \kappa_c$, which presumably allows only a trivial continuum limit [5.41–5.43]. A separatrix GM of the flow lines extends into the interior of the flow diagram, in accord with the expected line of phase transitions. This close correspondence between the flow and phase diagrams is reassuring, for such detailed agreement (unlike the case with critical exponents) is not guaranteed by universality. If this separatrix actually represents a phase transition, then the point M at which it terminates must be a fixed point of the renormalization group transformation. This fixed point is *marginal* ($y_1 = 0$) in the direction $\delta\mu_1$ of the separatrix connecting it to point G, since the renormalization group flows go in the same direction on both ($\pm\delta\mu_1$) sides of the fixed point. The critical exponent y_2 which governs this phase transition can be determined by measuring the derivatives of the renormalization group flows in the direction $\delta\mu_2$ *orthogonal* to this separatrix. If, for example, the separatrix depicts a first-order phase transition, then according to the standard lore y_2 equals four [5.53]. The matrix of derivatives of the renormalized couplings with respect to the original couplings, $\partial K'_a/\partial K_b$, has eigenvalues b^{y_1} and b^{y_2}, where b is the scale factor of the blocking transformation. Unfortunately, these eigenvalues have yet to be directly measured.

Other inferences can also be drawn from the topology of the flow diagram. For example, the presence of a marginal fixed point generally implies the existence of logarithmic corrections to scaling (in this case along the separatrix MG) [5.35, 5.47]. Should the segment MG turn out to be a line of first-order phase transitions, it is unlikely that a continuum theory can be defined there (except possibly at the point M itself). Therefore, given the likelihood of that possibility, we return to the full lattice standard model in search of a continuum theory.

5.4.3. *Flow structure of the lattice standard model*

The lattice standard model can be treated by techniques similar to those used in the SU(2)–Higgs model discussed above. Many analytical results (including the phase diagram) for this model have been reported [5.34, 5.54]. One Monte Carlo renormalization group study has also been performed [5.35]. Some of the results of these analyses will now be described.

Like the SU(2)–Higgs studies described above, the renormalization group flow structure presented here (in fig. 5.2) for the lattice standard model is the result of a maximally truncated calculation. A set of three coupled linear equations is generated by using the invariance of the integration measure under infinitesimal gauge rotations, coupled with the assumption that the block action is well approximated by the functional form [cf. eq. (5.34)]

$$S'\{U', V', \Phi'\} = \beta_1' S_1\{U'\} + \beta_2' S_2\{V'\} + \kappa' S_L\{U', V', \Phi'\} . \tag{5.50}$$

The equations used to generate the block renormalized couplings are determined from the SU(2) invariance of the integration measure [cf. eqs. (5.46)],

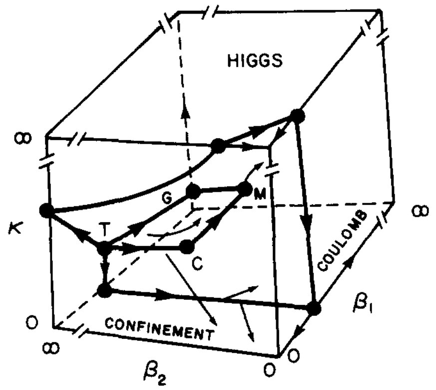

Fig. 5.2. Flow diagram of the lattice standard model.

$$\mathrm{Tr}\, \boldsymbol{\Delta}_{n'} \cdot [(\boldsymbol{\Delta}_{n'} S'_{\mathrm{L}})\, \mathrm{e}^{-S'}] = 0\,, \tag{5.51a}$$

$$\mathrm{Tr}\, \boldsymbol{\Delta}_{n',\mu} \cdot [(\boldsymbol{\Delta}_{n',\mu} S'_2)\, \mathrm{e}^{-S'}] = 0\,, \tag{5.51b}$$

and the U(1) invariance of this measure,

$$\mathrm{Tr}\, \frac{\delta}{\delta\theta'_{n,\mu}} \left[\left(\frac{\delta}{\delta\theta'_{n,\mu}}\, S'_1 \right) \mathrm{e}^{-S'} \right] = 0\,, \tag{5.51c}$$

where the definitions

$$U'_{n,\mu} \equiv \exp(\mathrm{i}\theta_{n,\mu})\,, \qquad \mathrm{Tr} \equiv \int \mathrm{D}U \int \mathrm{D}V \int \mathrm{D}\Phi^+ \int \mathrm{D}\Phi \tag{5.52}$$

have been used. Equations (5.51) yield a coupled set of three equations for the three unknown block couplings β'_1, β'_2 and κ'.

The flow diagram generated by such methods is likely to be a crude approximation to the true renormalization group flow structure of the theory. It can, however, be a useful guide into unknown territory, for the actual location of the continuum limit of a lattice theory is not *a priori* obvious from its phase structure. Despite the uncertainties involved, such a preliminary analysis can be useful, if only to produce a *conjecture* as to the location and nature of the continuum limit(s). Moreover, the conjecture thus produced can be systematically improved by the inclusion of more operators in the truncation scheme.

Figure 5.2 shows the phase structure of this theory [5.34]. Note that there are three phases (described above), labelled "confinement", "Coulomb", and "Higgs", respectively. The confinement phase is distinct from the Coulomb and Higgs phases, which are analytically connected.

The detailed flow structure generated in the maximal truncation scheme [5.35], like the detailed phase structure [5.34], is quite complex. Thus only a survey of the most important features is presented here. In particular, only two of the several additional fixed points which appear in the full theory are depicted in fig. 5.2. These are the two fixed points with the greatest number of relevant directions (labelled "C" and "T" in fig. 5.2).

One salient feature of the flow diagram is that no fixed point appears at finite β_2 with *three* relevant directions. The most relevant fixed point at finite β_2, labelled "C" in fig. 5.2, has one relevant direction (essentially along the β_1 axis) and two *marginal* directions (essentially along the κ and β_2 axes). Should a nontrivial quantum field theory exist at this point, two of the renormalized parameters of the theory should be *bounded* (recall from section 3 that marginal directions in general correspond to bounds on couplings) and the remaining one unconstrained.

Another fixed point occurs at infinite β_2. This fixed point is labelled "T" in fig. 5.2. The values of κ and β_1 are finite at this fixed point. This combination suggests that a continuum theory defined at this point is asymptotically free in the SU(2) gauge coupling, i.e., is defined at zero g_2 (infinite β_2). This in turn suggests that if a nontrivial quantum theory exists at this point, then the SU(2) gauge fields may not be necessary for the theory to be nontrivial.

The efficacy of the renormalization group is now evident. The search for the most likely locations in the lattice parameter space for a continuum limit for the theory can be narrowed in scope. Moreover, predictions can be made about the number and range of the independent renormalized couplings in the continuum theory. Thus, for example, if (as is suggested by the universality hypothesis) the quartic coupling constant is irrelevant at the fixed point then one of the parameters of the theory (e.g. the Higgs mass) can be *predicted* from the others [5.13, 5.35].

Several important studies must be done, however, before any firm conclusions can be drawn. For example, little is known about the variable-magnitude $SU(2) \times U(1)$ lattice standard model [5.55]. Moreover, no comprehensive analysis of the effects of variant or higher-representation lattice actions in this model has been made [5.58]. Further computations (by the Monte Carlo renormalization group or other methods like finite-size scaling [5.57]) are needed before the critical exponents of the theory can be obtained. These exponents would yield the anomalous dimensions of various operators (like Φ^2) in the theory, and could be used to learn whether a nontrivial continuum limit for the standard model is possible. Much work therefore remains to be done, and the fate of the standard model as a fundamental theory still remains an open question.

5.5. *The standard model as a low-energy effective theory: What if there is no fundamental Higgs particle?*

The preceding subsection has made it clear that the standard model might be a trivial theory, although the issue is far from settled. Should this triviality be demonstrable, it is fair to question the whole idea of symmetry breaking by elementary scalars, and, *ipso facto*, the necessity of their existence.

Yet the concept of spontaneous symmetry breaking by elementary scalar Higgs particles may still be a metaphor of some utility in theories of the weak interaction. Even if Higgs particles do not exist, it is worthwhile to ask whether the standard model can be used as an effective phenomenological theory up to a certain cutoff momentum scale.

Ideas of this nature are certainly not new. They are implicit in early estimates [5.58] of upper bounds on the mass of the Higgs boson. These bounds are predicated upon the assumption that tree-level unitarity must be respected unless "new physics" appears or the weak interaction becomes "strong" at some scale. (The idea of a Higgs sector which is strongly interacting and related ideas have been considered by several authors [5.59]; however, this possibility may be generally inconsistent with the idea that a pure Higgs theory is trivial.)

The concept that the standard model is an effective theory, valid up to some scale at which unspecified "new physics" appears, is rather vague, but gains credence from a well-known "decoupling" theorem [5.50]. According to this theorem, the physical effects of high-mass particles in renormalizable field theories are unobservable at low energies. Various attempts to estimate the scale at which the "new physics" appears (or, conversely, to bound standard model parameters under the assumption that no "new physics" appears up to some specified large momentum scale) have been made [5.61, 5.13, 5.62–5.66, 5.41, 5.43]. A brief summary of a few of these ideas is now made.

5.5.1. *Upper bounds on the Higgs mass from triviality: General philosophy*

The assumption that a pure ϕ^4 field theory is trivial can lead to bounds of phenomenological relevance. Specifically, it is assumed that the standard model is inconsistent as a fundamental theory but is a good effective theory up to some momentum scale Λ. Then the demand that the Landau ghosts predicted by naive perturbation theory be avoided generates restrictions on the renormalized couplings in the theory.

The simplest approach is to assume that gauge fields do not significantly modify the beta function for the quartic coupling. From the discussion of subsection 5.2, it is clear that this is a very strong (and probably invalid) assumption, yet the results are conceptually quite clear. Consider a pure Φ^4 field theory defined by the Lagrange density

$$L_{\mathrm{H}} = \tfrac{1}{2}(\partial_\mu \Phi)^{+}(\partial^\mu \Phi) - \tfrac{1}{2}m_0^2 \Phi^{+}\Phi - \tfrac{1}{4}\lambda_0(\Phi^{+}\Phi)^2 , \qquad (5.53a)$$

where Φ is the usual SU(2) complex doublet. The beta function for this theory to one-loop order is

$$\beta(\lambda) = \frac{d\lambda}{dt} \simeq \frac{3}{2\pi^2} \lambda^2 \tag{5.53b}$$

by the results of subsection 5.2. The solution of eqs. (5.53) is given by

$$\frac{1}{\lambda(t)} = \frac{1}{\lambda_R} - \frac{3}{2\pi^2} t , \tag{5.54a}$$

where in terms of the running momentum scale Q and renormalization point μ,

$$t = \ln(Q/\mu) , \tag{5.54b}$$

and λ_R is the renormalized ($t = 0$) quartic coupling. For a given λ_R the running coupling $\lambda(t)$ becomes negative at some scale Λ. Thus $\Lambda(\lambda_R)$ is the largest scale at which the theory makes sense (in the one-loop approximation). Conversely, if Λ denotes the largest momentum scale at which the theory is still valid, then an upper bound on the renormalized quartic coupling λ_R is implied [5.62],

$$\lambda_R \leq \frac{1}{(3/2\pi^2)\ln(\Lambda/\mu)} . \tag{5.55}$$

From the tree-level result for the ratio of Higgs mass to W mass in the standard model,

$$m_H/m_W = (8\lambda_R/g^2)^{1/2} , \tag{5.56}$$

an upper bound on the Higgs mass is implied [5.62],

$$m_H \leq m_{H,\max} = m_W \frac{4\pi}{g\sqrt{3}} (\ln \Lambda/\mu)^{-1/2} = 900\ \text{GeV} (\ln \Lambda/\mu)^{-1/2} , \tag{5.57}$$

which suggests a conservative upper bound of about 1 TeV for the Higgs boson mass, since Λ must be larger than m_H.

Obviously the bound (5.57) requires in addition not only the assumption that the gauge and Yukawa couplings remain unimportant up to the scale Λ, but also the assumption that higher-order terms in λ in the beta function (5.53) are likewise insignificant. This latter assumption may be obviated by numerical simulation of a lattice model, as was suggested in ref. [5.62]. Numerical simulations of this nature have recently been performed [5.63, 5.43], and imply that renormalized perturbation theory gives essentially the correct results, provided that the "cutoff" scale Λ is reasonably large.

5.5.2. *Upper bounds on the Higgs mass from coupled gauge–Higgs systems*

As discussed in the above subsection, the addition of gauge fields can significantly change a pure ϕ^4 theory. For example, a field theory with gauge fields coupled to elementary scalars can be perturbatively asymptotically free in all couplings even though the pure scalar theory is not. Thus it is not surprising that the addition of gauge fields can improve the bound eq. (5.57), as was pointed out in various contexts [5.13, 5.64, 5.62]. These bounds have since been elaborated upon by numerous authors [5.65–5.67, 5.41, 5.43].

A general property of these bounds is the observation that the ratio $\lambda(t)/g_1^2(t)$ (the running quartic coupling to the running squared $U(1)$ coupling constant) should not become unreasonably large before the momentum cutoff Λ is reached. This seems to be a sensible minimal condition for consistency in a theory, for, if there is an energy regime where the running quartic coupling $\lambda(t)$ is much larger than the other couplings, a reasonable expectation is that the scalar sector will decouple from the rest of the theory. It is then difficult to see how the theory can escape triviality (but see ref. [5.59]).

These bounds [5.13] improve noticeably when the top quark Yukawa coupling is also included (see, e.g., ref. [5.64]). These authors identify three cases:

Case A. When the top quark mass m_t is less than 80 GeV, the authors of ref. [5.64] quote an upper bound $m_{H,max}$ on the Higgs mass of about 125 GeV. Obviously this bound is remarkably low.

Case B. When $m_t > 80$ GeV, their bounds on the Higgs mass vary sharply with m_t. The bounds $65\,\text{GeV} < m_H < 122\,\text{GeV}$ for $m_t = 120$ GeV and $140\,\text{GeV} < m_H < 148\,\text{GeV}$ for $m_t = 150$ GeV are reported.

Case C. For $m_t = 168$ GeV, the largest value allowed in this scenario, $m_H \simeq 175$ GeV. Note that this value is a *prediction*, and not a bound.

The complexity and importance of this issue demands that full nonperturbative studies of the problem be performed [5.35]; nevertheless it is clear that the problem is interesting and worthy of further analysis. It may even be that a fundamental cutoff is required in the theory [5.66, 5.68], although such a pronunciamento requires an understanding of the role of the axial anomaly [5.69] in a cutoff theory. Naively, this anomaly vanishes when a cutoff is introduced [5.36], and implies, for example, that the $\pi^0 \to 2\gamma$ decay rate is zero. (A regularization scheme which respects chiral symmetry is assumed here.)

Thus it can be seen that, with the addition of certain specific assumptions, the conjecture that the minimal standard model is a good effective model (but *not* a consistent fundamental theory) up to a large momentum scale Λ implies a bound on the mass of the Higgs particle. Experimental measurements up to a mass of about $m_{H,max}$ in this scenario should thus either find the Higgs particle, or some unspecified "new physics" which causes the formalism to break down.

6. Will Higgs particles ever be found?

> "*I would like to offer a theoretical prediction at the 5% confidence level: within five years there will be a rigorous construction of the solutions of $\lambda(\phi^4)_4$ and spin-$\frac{1}{2}$ quantum electrodynamics in four-dimensional space-time.*"
>
> Arthur S. Wightman (1977) [6.1]

> "*When we try to pick out anything by itself, we find it hitched to everything else in the universe.*"
>
> John Muir

Expectations often remain unfulfilled. Much of the early excitement over the standard model of the weak interactions was caused by the discovery that it was renormalizable, and hence could be considered a viable candidate for a fundamental theory of elementary particles. The standard model was a remarkable achievement, for it included the *lucus a non lucendo* of massive gauge bosons in a consistent fashion and so afforded a place for the mediators of the weak force, the $W^{\pm}$ and Z^0. The standard model also posits the existence of an unobserved elementary scalar particle, the Higgs boson. The *interactions* of this Higgs particle are a necessary ingredient in the standard model.

As discussed in sections 2–4 of this review, strong evidence suggests that a theory which only includes Higgs particles is "trivial" or *noninteracting*. Should this triviality persist when the Higgs is "hitched to everything else" in the standard model, then the standard model scenario is in trouble. It would then be fair to ask if a real "Higgs particle" should exist, or whether this putative elementary scalar is instead just an invisible metaphor for a more complicated mechanism. The obvious question to ask—what this new mechanism may be—is not as obviously answerable.

A more informative set of possibilities does, however, exist if the Higgs sector of the standard model (or indeed of any realistic Higgs model) is nontrivial. In this case it is likely possible to calculate upper bounds upon the Higgs mass or even to predict its value. The requirement that the theory be nontrivial thus can imply phenomenological constraints on the theory (see section 5). By contrast, a naive semiclassical analysis of the scalar sector of the standard model does *not* yield any information on the Higgs mass.

It is evident that the question of the triviality of Higgs models is of more than philosophical interest. The inquiry as to whether an elementary scalar particle exists (and if its mass is predictable) is also given immediacy by the contemporary agenda for the construction of new accelerators. The intended purpose of these machines typically includes a search for the Higgs particle. Fortunately for these experimental efforts, progress is being made in the understanding of triviality and its implications for elementary particle phenomenology. Much remains to be done, however. Although techniques such as the analysis of lattice gauge theories by the Monte Carlo renormalization group may well hold the key to the riddle of triviality, the final answers are not yet known. This report therefore concludes with the same question as it started with: Can elementary scalar particles exist? The future holds the answer.

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